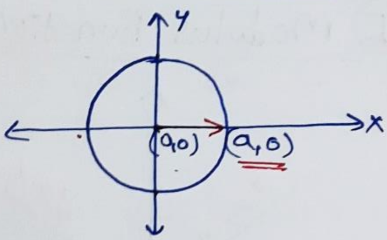
Application of Integrals [समाकलनों के अनुप्रयोग] Area

- Some Graphs
- Some Integration Formulas
- Concept: $\int_a^b f(x)\,dx$ → + Area, − Area

Some Graphs:

(I) Circle

$\boxed{x^2+y^2=a^2}$



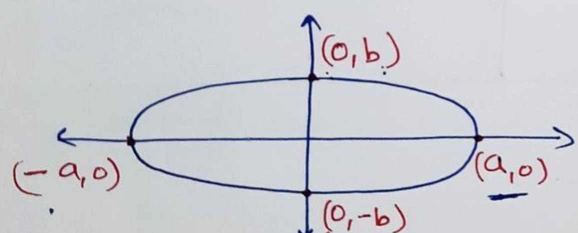
Centre (0,0), radius = a

$\boxed{(x-h)^2+(y-k)^2=a^2}$

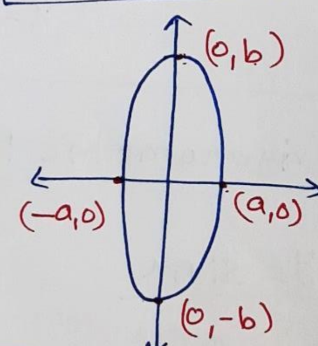
centre (h,k)
radius = a

(II) Ellipse

$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $(a>b)$

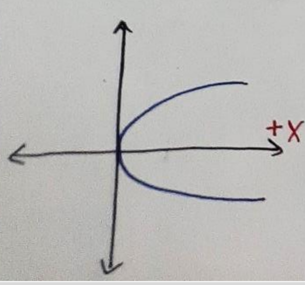


$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $(a<b)$

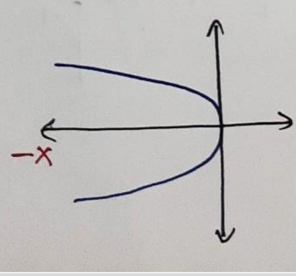


(III) Parabola

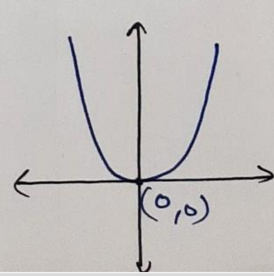
$\boxed{y^2=4ax}$



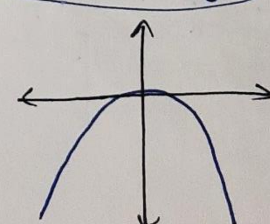
$\boxed{y^2=-4ax}$



$x^2=4ay$



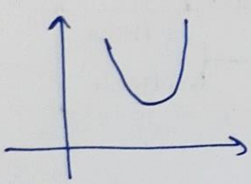
$x^2=-4ay$



Parabola

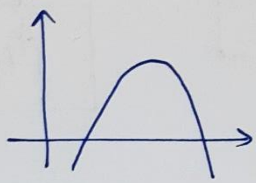
$\boxed{y = ax^2 + bx + c}$

$\boxed{a>0}$



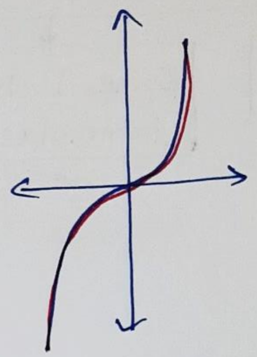
Upward Parabola

$\boxed{a<0}$

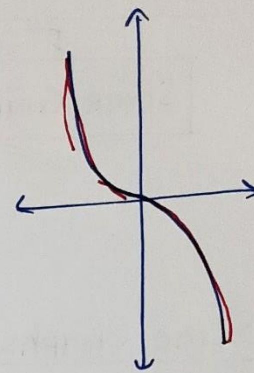


Downward Parabola

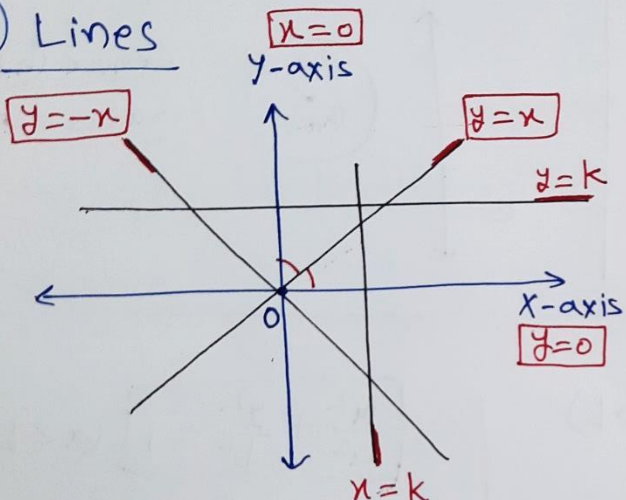
$y = x^3$



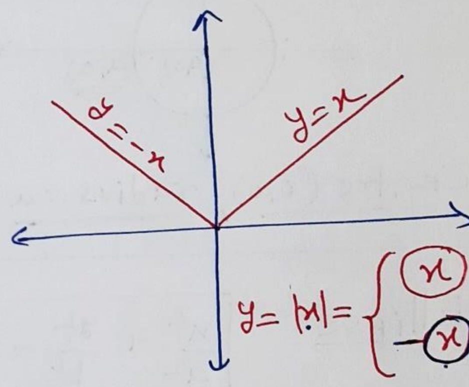
$y = -x^3$



IV Lines



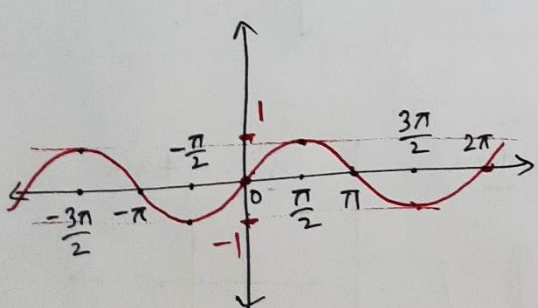
V Modulus Function: $\boxed{y=|x|}$



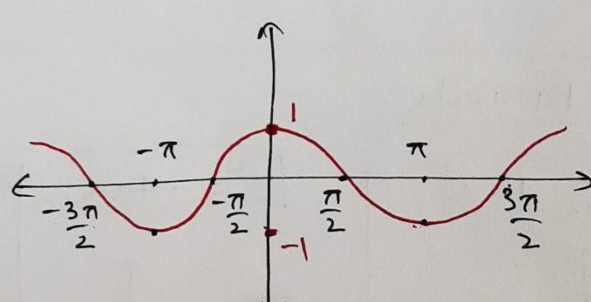
$$y = |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$

VI Trigonometric Functions

$y = \sin x$



$y = \cos x$



Some Integration Formulas

$\int 1 \cdot dx = x + c$

$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$

$\int x \cdot dx = \frac{x^2}{2} + c$

$\int \sin x \cdot dx = -\cos x + c$

$\int x^2 \cdot dx = \frac{x^3}{3} + c$

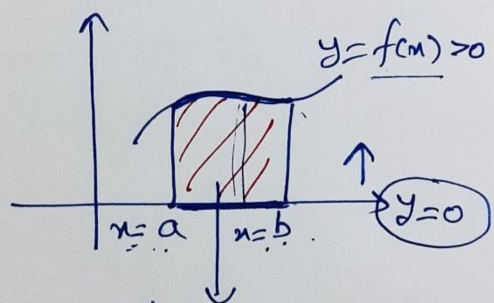
$\int \cos x \cdot dx = \sin x + c$

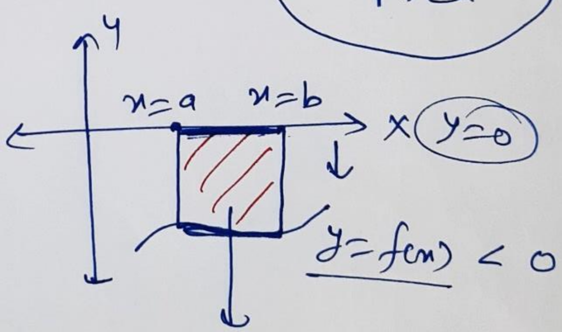
$\int \sqrt{x} \cdot dx = \frac{2}{3} x^{3/2} + c$

$$\boxed{\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

Concept (Application of Integrals)

Meaning of Definite Integrals = Algebraic Area ??





$\text{Area} = \int_a^b f(x) \cdot dx = \oplus ve$

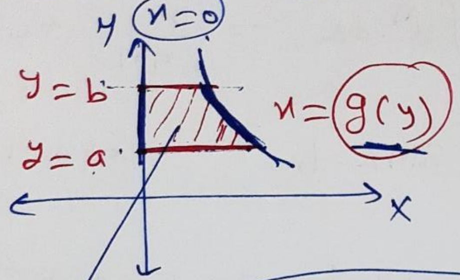
$\text{Area} = \left|\int_a^b f(x) \cdot dx\right| = |\ominus ve| = \oplus$

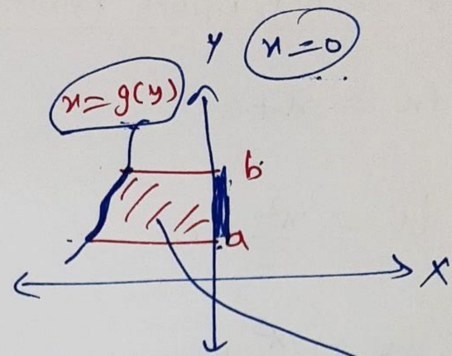
$$A = \boxed{\int_{x=a}^{x=b} (\text{उपर वाला Graph} - \text{नीचे वाला Graph}) \cdot dx} = \oplus ve \text{ Always}$$

$$\boxed{\text{Area} = \int_a^b (f(x) - 0) \cdot dx = \oplus}$$

$\text{Area} = \int_a^b (0 - f(x)) \cdot dx = \oplus ve$

with respect to 'y'





$$\text{Area} = \int_{y=a}^{y=b} \Big(\overset{\text{दांया}}{\underline{\text{Right वाला}}} - \overset{\text{बांया}}{\underline{\text{Left वाला}}}\Big)\,.\,dy$$

y की terms

$$\text{Area} = \int_a^b \big(g(y) - 0\big)\,.\,dy$$

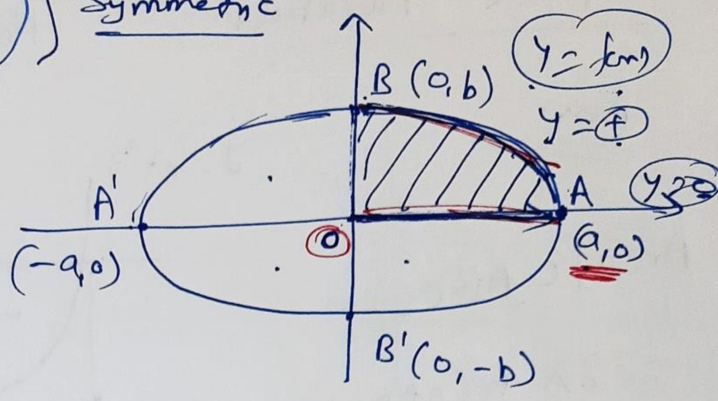
$$= (+)\text{ve.}$$

$$\boxed{\text{Area} = \int_a^b \big(0 - g(y)\big)\,.\,dy}$$

e.g. Find the area enclosed by the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ans.

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ } Symmetric



Ellipse Area (ABA'B'A)

$= 4 \times \boxed{ar(OABO)}$

$= 4 \times \int_{x=0}^{x=a} (y - 0) \cdot dx$

Curve x-axis

(ऊपर − नीचे)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$\Rightarrow \frac{y}{b} = \pm\sqrt{\frac{a^2-x^2}{a^2}}$$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

+ − term sin x

$$= 4 \times \int_0^a \frac{b}{a}\sqrt{a^2-x^2} \cdot dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2} \cdot dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) - 0 - 0\right]$$

$\rightarrow 0$

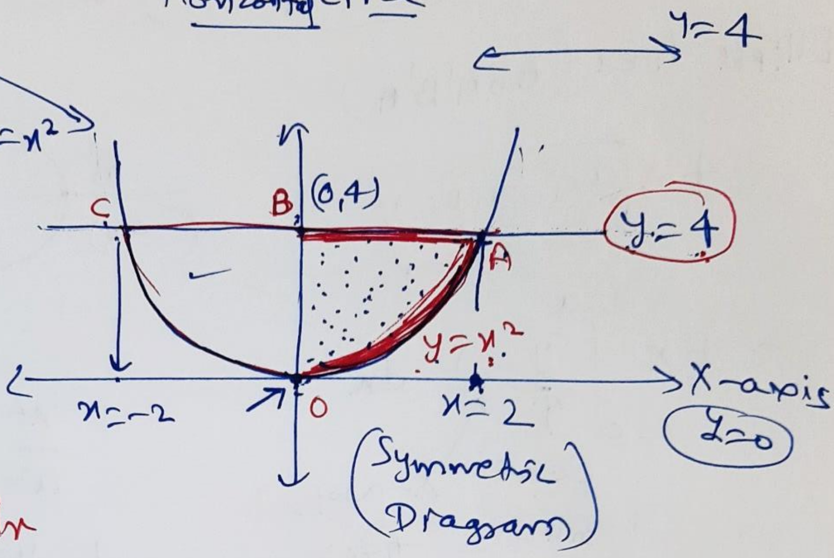
$$= \frac{\cancel{4}^{2} b}{\cancel{a}} \times \frac{a^{\cancel{2}}}{\cancel{2}} \cdot \sin^{-1}(1)$$

$\sin\frac{\pi}{2} = 1$

$$= \cancel{2}\, b a \frac{\pi}{\cancel{2}} = \pi a b$$

e.g. Find the area of the region bounded by the curve $\underbrace{y = x^2}_{\text{upward Parabola}}$ and line $\underbrace{y = 4}_{\text{Horizontal Line}}$.

Ans.

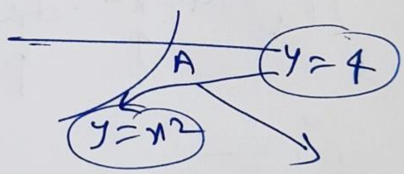


Area (OABCO)

$= 2 \times ar(OABO)$

$$= 2 \times \int_{x=0}^{x=2} \underbrace{4}_{\substack{\text{line} \\ (\text{ऊपर})}} - \underbrace{x^2}_{\substack{\text{curve} \\ (\text{नीचे})}} \cdot dx$$

for U.L. = 'A' का X-coordinate



$4 = x^2$

$x = \pm 2$

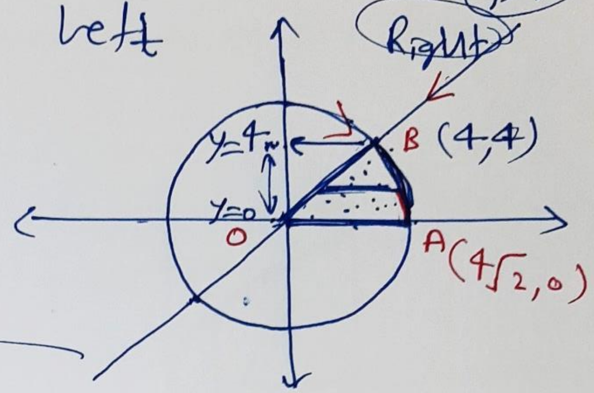
$$= 2 \times \left(4x - \frac{x^3}{3}\right)_0^2$$

$$= 2\left(\underbrace{8 - \frac{8}{3}}_{} - \underbrace{0}_{}\right)$$

$$= 2 \times \frac{16}{3} = \frac{32}{3}$$ ~~Sq~~ unit squares

(+) ✓

e.g. Find the area of the region in the first quadrant enclosed by the x-axis, line $y=x$, circle $x^2+y^2=32$.



Ans. Circle $x^2+y^2=32$

Centre $=(0,0)$

$r=\sqrt{32}=\sqrt{16\times2}=4\sqrt{2}$

$$\int_{y=\text{L.L.}=0}^{y=\text{U.L.}=4}(\text{दायाँ}-\text{बायाँ})\cdot dy$$

circle line

$$ar(OABO)=\int_0^4(\text{Curve}-\text{line})\cdot dy$$

y की terms

$$=\int_0^4\Big(\underbrace{\sqrt{32-y^2}}_{\substack{x\text{-Curve} \\ (\text{दायाँ})}}-\underbrace{y}_{\substack{\text{line } x=y \\ (\text{बायाँ})}}\Big)\cdot dy$$

$$=\int_0^4\left(\sqrt{(4\sqrt{2})^2-y^2}-y\right)\cdot dy$$

$$=\left[\frac{y}{2}\sqrt{32-y^2}+\frac{32}{2}\sin^{-1}\left(\frac{y}{4\sqrt{2}}\right)-\frac{y^2}{2}\right]_0^4$$

$$=4\pi$$

Point of intersection B

(4, 4)

$x^2+y^2=32$ —

$y=x$ —

$x^2+x^2=32$

$\cancel{2}x^2=\cancel{32}\ 16$

$x=\pm4$

$x^2+y^2=32$

$x^2=32-y^2$

$x=\pm\sqrt{32-y^2}$

$y=x$

$x=y$

Exercise 8.1

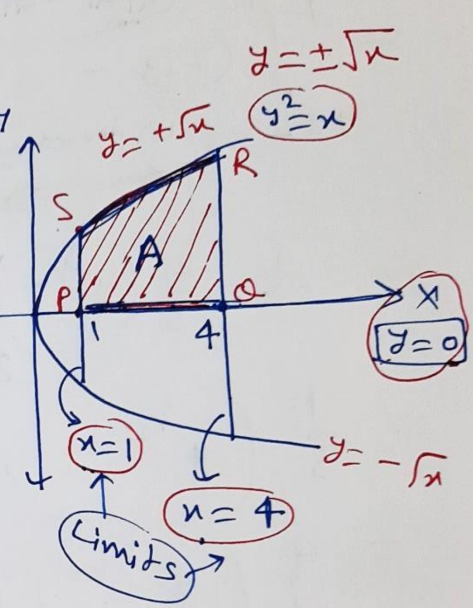
Q.1 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x=1$, $x=4$ and the x-axis in the first quadrant.

vertical lines.

Ans. $y = \sqrt{x}$

curve $\qquad$ (x-axis)

(ऊपर वाला – नीचे वाला)



$$ar(PQRSP) = \int_{x=1}^{x=4} \underbrace{(\sqrt{x} - 0)}_{\text{in terms of } x}.dx$$

$$= \int_1^4 \sqrt{x}\,.dx$$

$$= \frac{2}{3}\left(x^{3/2}\right)_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1)$$

$$= \frac{14}{3} \text{ Sq. units.}$$

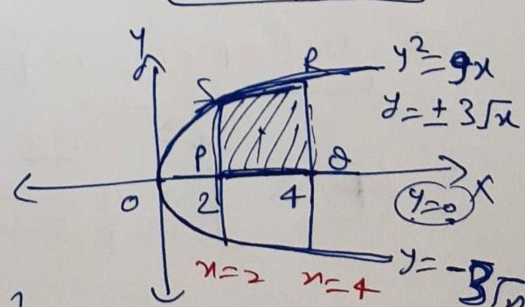
Q.2 Find the area of the region ~~bou~~ bounded by $y^2 = 9x$, $x=2$, $x=4$ and the x-axis in the first quadrant.

Vertical

Ans.

ऊपर $\quad$ नीचे

$$ar(PQRSP) = \int_2^4 \underbrace{(3\sqrt{x} - 0)}_{\text{terms in } x}.dx$$

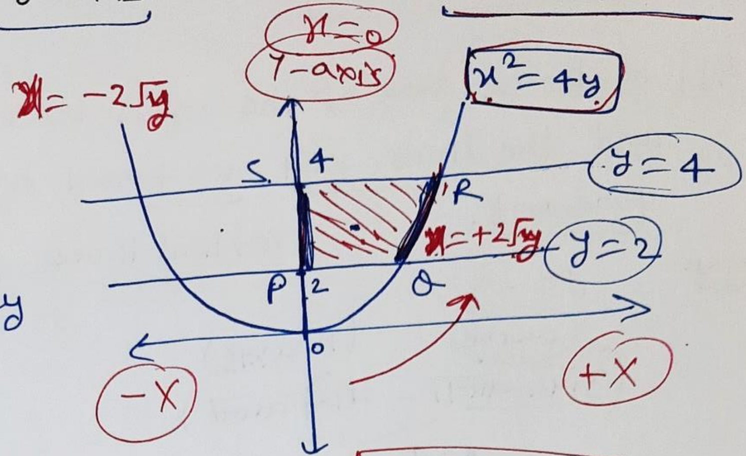


$$= \cancel{3}.\frac{2}{\cancel{3}}\left(x^{3/2}\right)_2^4 = 2\left[4^{3/2} - 2^{3/2}\right]$$

$$= 2(8 - 2\sqrt{2}) = (16 - 4\sqrt{2})$$

Q.3 Find the area of the region bounded by $x^2 = 4y$, $y=2$, $y=4$ and the y-axis in the first quadrant.

Ans. Horizontal,



$ar(PQRSP)$ दांया − बांया

$$= \int_{y=2}^{y=4} (2\sqrt{y} - 0)\,dy$$

in terms of (y)

$$\boxed{\begin{array}{l} x^2 = 4y \\ x = \pm 2\sqrt{y} \end{array}}$$

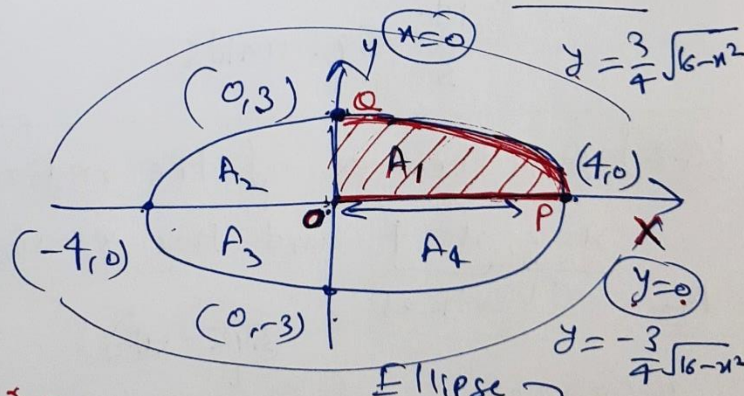
$$= 2 \cdot \frac{2}{3}\left(y^{3/2}\right)_2^4$$

$$= \frac{4}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{4}{3}\left(8 - 2\sqrt{2}\right) = \frac{32 - 8\sqrt{2}}{3}$$ Sq. units.

Q.4 Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Ans. $A_1 = A_2 = A_3 = A_4$



Area of ellipse

$= ar(OPQO) \times 4$

(ऊपर − नीचे)

Ellipse ⟹ Symmetrical

$$= 4 \times \int_0^4 \left(\frac{3}{4}\sqrt{16 - x^2} - 0\right).dx$$

terms of (x)

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow \frac{y^2}{9} = 1 - \frac{x^2}{16}$$

$$\Rightarrow \frac{y}{3} = \pm\sqrt{\frac{16 - x^2}{16}} \Rightarrow y = \pm\frac{3}{4}\sqrt{16 - x^2}$$

$= \cancel{4} \times \frac{3}{\cancel{4}} \int_0^4 \sqrt{16-x^2} \cdot dx$ $\quad \underline{a^2 = 16} \quad a = 4$

$$\boxed{\int \sqrt{a^2-x^2} \cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$= 3\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$

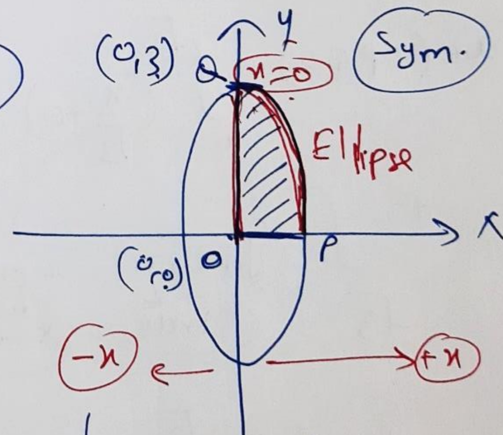
$= 3\left[\cancel{\frac{4}{2}\sqrt{16-16}}^{\,0} + 8\sin^{-1}(1) - 0\right]$

$= 3 \times \cancel{8}^{4} \times \frac{\pi}{\cancel{2}}$

$= 12\pi$

Q.5 Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$



area of ellipse $= 4 \times ar(OPQO)$

$= 4 \times \int_0^3 \left(\frac{2}{3}\sqrt{9-y^2} - 0\right) \cdot dy$

terms of (y)

(दांया − बांया)

$= \frac{8}{3}\int_0^3 \sqrt{9-y^2} \cdot dy$

$= \frac{8}{3}\left[\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right]_0^3$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\Rightarrow \frac{x^2}{4} = 1 - \frac{y^2}{9}$

$\Rightarrow x^2 = 4\left(\frac{9-y^2}{9}\right)$

$\Rightarrow x = \pm\frac{2}{3}\sqrt{9-y^2}$ (+)

$$\text{area of ellipse} = \frac{8}{3}\left(\frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right]_0^3$$

$$= \frac{8}{3}\left(\frac{3}{2}\cancel{\sqrt{9-9}}^{\,0} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - 0\right)$$

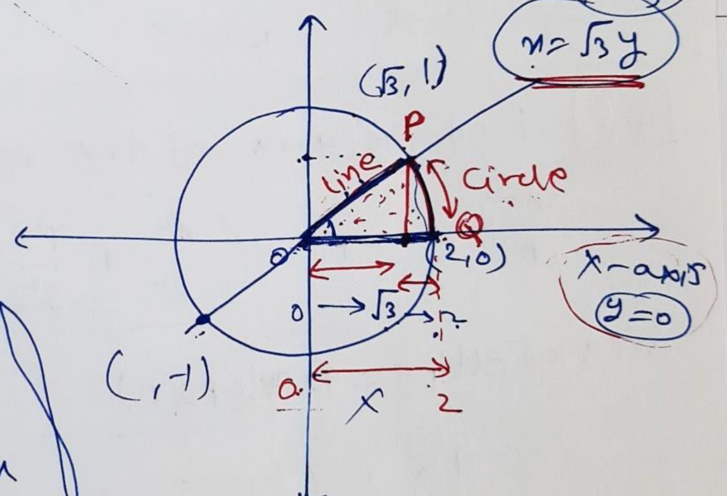
$$= \frac{\cancel{8}^{\,4}}{\cancel{3}} \times \frac{\cancel{9}^{\,3}}{\cancel{2}}\sin^{-1}(1) = 12 \times \frac{\pi}{2} = 6\pi \text{ sq. units.}$$

Q.6 Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}\,y$ and the circle $x^2+y^2=4$.

Ans.

Centre = $(0,0)$

$r = 2$



$$ar(OPQO) = \int_0^{\sqrt{3}}\left(\frac{x}{\sqrt{3}} - 0\right).dx$$

$$+ \int_{\sqrt{3}}^{2}\left(\sqrt{4-x^2} - 0\right).dx$$

Circle   x-axis

P (Point of intersection of $x^2+y^2=4$ & $x=\sqrt{3}y$)

$(\sqrt{3}y)^2 + y^2 = 4$

$3y^2 + y^2 = 4$

$4y^2 = 4 \Rightarrow y^2 = 1$

$y = \pm 1$

$y = 1$

$x = \sqrt{3}$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}$$

$$+ \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$ar(OPQO) = \underbrace{\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\cancel{\sqrt{3}}}\left(\frac{\overset{\sqrt{3}}{\cancel{3}}}{2} - 0\right) + \left[\cancel{\frac{x}{2}\sqrt{4-4}}^{\,0} + 2\sin^{-1}(1) - \frac{\sqrt{3}}{2}\sqrt{4-3} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

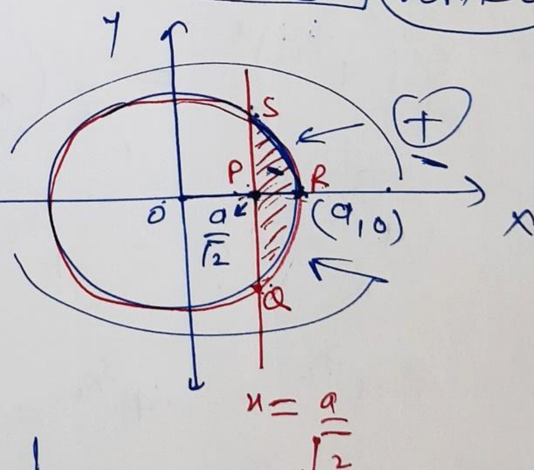
$$= \cancel{\frac{\sqrt{3}}{2}} + \cancel{2}\times\frac{\pi}{\cancel{2}} - \cancel{\frac{\sqrt{3}}{2}} - 2\times\frac{\pi}{3}$$

$$= \pi - \frac{2\pi}{3} = \frac{3\pi - 2\pi}{3} = \frac{\pi}{3}$$

Q.7 Find the area of the smaller part of the circle $\underbrace{x^2+y^2=a^2}$ cut off by the line $\underbrace{x = \frac{a}{\sqrt{2}}}$. Vertical

Ans. Centre $(0,0)$, $r = a$

$ar(PRSP) = ar(PRQP)$



$$ar(PQRSP) = 2\times ar(PRSP)$$

$$= 2\times\int_{\frac{a}{\sqrt{2}}}^{a}\left(\underbrace{\sqrt{a^2-x^2}}_{\text{Circle}} - \underbrace{0}_{y=0}\right)dx$$

$$= 2\times\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$x^2+y^2=a^2$

$y = \pm\sqrt{a^2-x^2}$

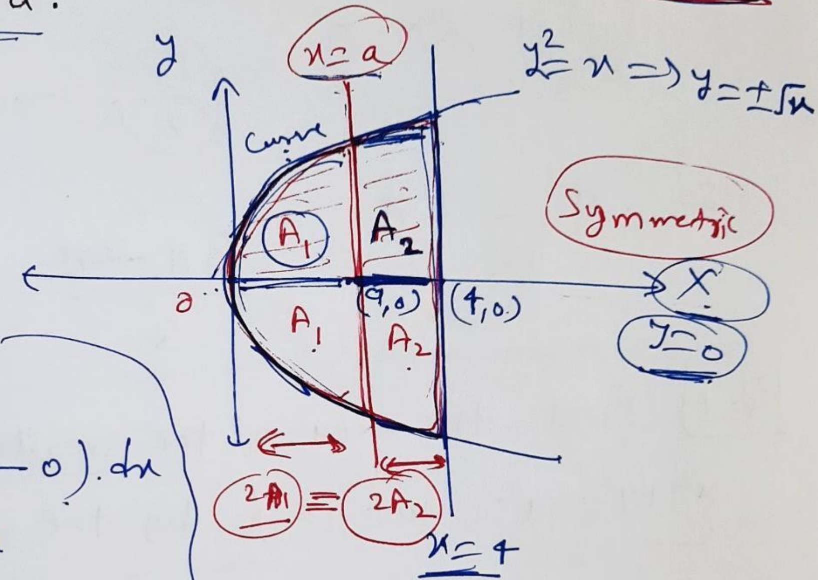
$$= 2\times\left[\cancel{\frac{a}{2}\sqrt{a^2-a^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}(1) - \frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

Exercise 8.1

Q.8 The area between $x=y^2$ and $x=4$ is divided into two equal parts by the line $x=a$, find the value of a.

Vertical

Ans.



Given $2A_1 = 2A_2$

$$A_1 = A_2$$

$$\int_0^a (\sqrt{x} - 0)\,dx = \int_a^4 (\sqrt{x} - 0)\,dx$$

Curve − Axis $\quad$ Curve

$$\Rightarrow \frac{2}{3}\left(x^{3/2}\right)_0^a = \frac{2}{3}\left(x^{3/2}\right)_a^4$$

$$\Rightarrow a^{3/2} - 0 = 4^{3/2} - a^{3/2}$$

$$\Rightarrow 2a^{3/2} = 4^{3/2} = \left(4^{1/2}\right)^3 = (2)^3 = 8$$

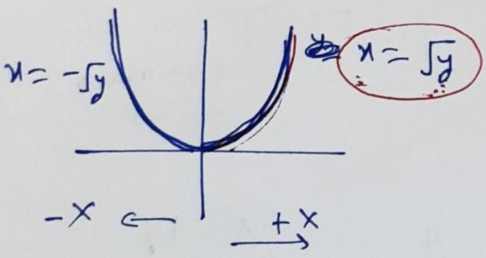
$$\Rightarrow \cancel{2}a^{3/2} = \cancel{8}\,4$$

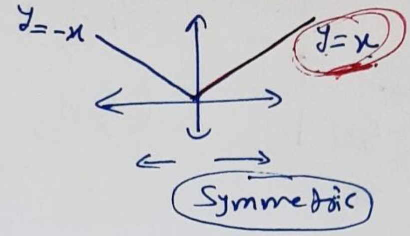
$$\Rightarrow a^{3/2} = 4$$

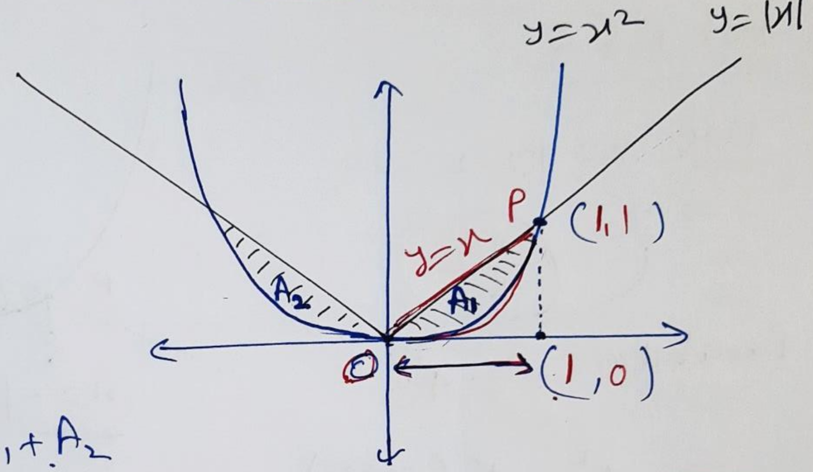
$$\Rightarrow \boxed{a = (4)^{2/3}}$$

Q.9 Find the area of the region bounded by the Parabola $y = x^2$ and $y = |x|$.

$\pm\sqrt{y} = x$







By Symmetry

$A_2 = A_1$

Required Area $= A_1 + A_2$

$= 2(A_1)$

terms in x

$$= 2 \cdot \int_{0}^{1} (x - x^2) \cdot dx$$

(ऊपर) − (नीचे)

Modulus $y = x$ Parabola $y = x^2$

P $y = x$, $x = \sqrt{y}$

$y = \sqrt{y}$

$\Rightarrow y^2 = y$

$\Rightarrow y^2 - y = 0$

$\Rightarrow y(y-1) = 0$

$y = 0, 1$

$x = 0, 1$

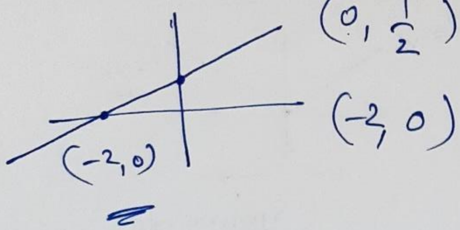
$(0,0)$ $(1,1)$

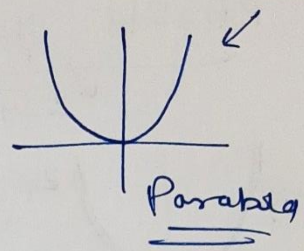
$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1$$

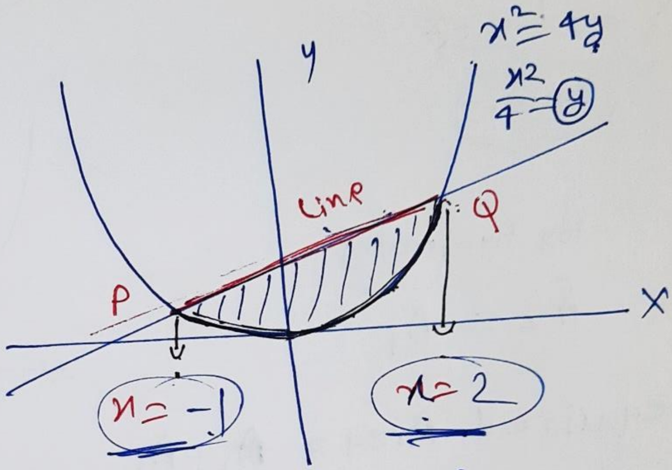
$$= 2\left(\frac{1}{2} - \frac{1}{3} - 0\right)$$

$$= \cancel{2}\left(\frac{3-2}{\cancel{6}_{3}}\right) = \frac{1}{3}$$

Q.10 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.







For P & Q

Line $x = 4y - 2$

$\Rightarrow y = \frac{x+2}{4}$

Parabola, $x^2 = 4y$

$\Rightarrow x^2 = \cancel{4}\left(\frac{x+2}{\cancel{4}}\right)$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow x(x-2) + 1(x-2) = 0$

$\Rightarrow (x-2)(x+1) = 0$

$x = 2, \ x = -1$

Required Area $= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right).dx$

Line Parabola

$= \frac{1}{4} \int_{-1}^{2} (x + 2 - x^2).dx$

$= \frac{1}{4} \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)_{-1}^{2}$

$= \frac{1}{4}\left(2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right)$

$5 - \frac{1}{2}$

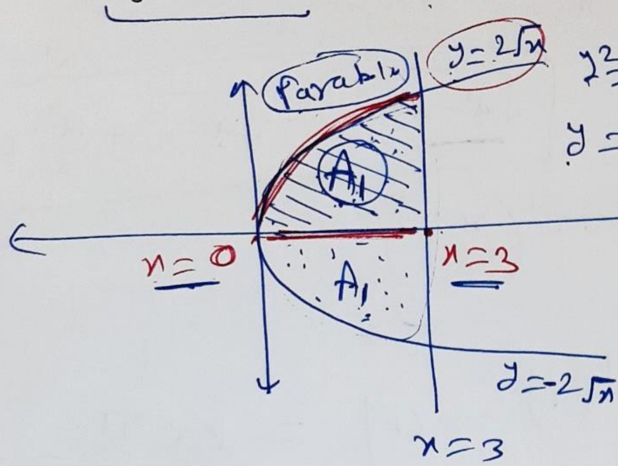
$\frac{9}{2}$

$= \frac{1}{4}\left(8 - \frac{1}{2} - 3\right)$

$= \frac{1}{4} \times \frac{9}{2} = \frac{9}{8}$

Q.11 Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Ans.



$y^2 = 4x$

$y = \pm\sqrt{4x} = \pm 2\sqrt{x}$

$y = 0$

Symmetric about X-axis

Required Area $= 2(A_1) = 2 \cdot \int_{x=0}^{x=3} (2\sqrt{x} - 0) \cdot dx$

(3742 − A1) Parabola

$= 4\int_0^3 \sqrt{x} \cdot dx$

$= 4 \cdot \frac{2}{3}\left(x^{3/2}\right)_0^3$

$= \frac{8}{3}\left(3^{3/2} - 0\right) = \frac{8}{3} \times 3^{1+\frac{1}{2}} = \frac{8}{\cancel{3}} \times \cancel{3} \cdot 3^{\frac{1}{2}}$

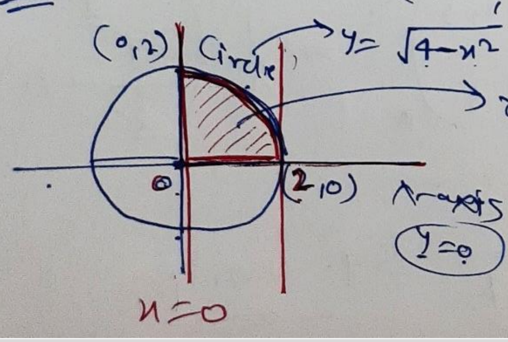
$= 8\sqrt{3}$

Q.12 Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and lines $x = 0$ & $x = 2$ is —

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Ans.

Centre $(0,0)$, $r = 2$



required Area $= \int_{x=0}^{x=2} (\sqrt{4-x^2} - 0) \cdot dx$

3742 − A1. Circle.

required area $= \int_{x=0}^{x=2} \sqrt{4-x^2} \, . dx$ $a^2 = 4$, $a = 2$

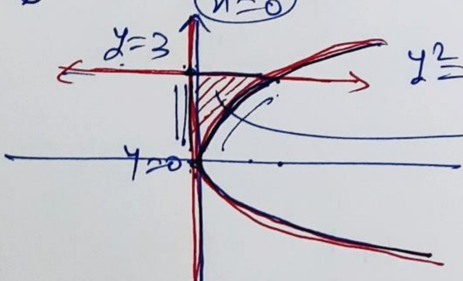
$$\boxed{\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \left[\frac{2}{2}\cancel{\sqrt{4-4}}^{\,0} + 2\sin^{-1}(1) - 0\right] = \cancel{2} \times \frac{\pi}{\cancel{2}} = \pi$$

Q.13 Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is —

(A) 2 (B) $\frac{9}{4}$ (C) $\frac{9}{3}$ (D) $\frac{9}{2}$



$y^2 = 4x \Rightarrow \frac{y^2}{4} = x$

The required area

$$= \int_0^3 \left(\frac{y^2}{4} - 0\right) . dy$$

Right − Left

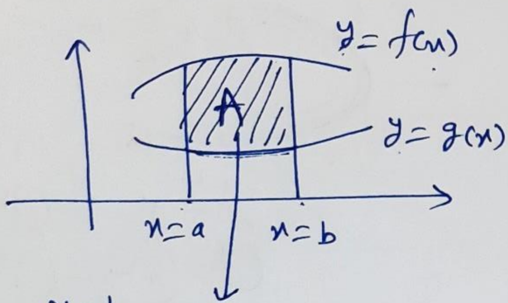
Parabola (y-axis) $x=0$

$$= \int_0^3 \frac{y^2}{4} \, . dy$$

$$= \frac{1}{4}\left(\frac{y^3}{3}\right)_0^3 = \frac{1}{4}\left(\frac{\cancel{27}^{\,9}}{\cancel{3}} - 0\right)$$

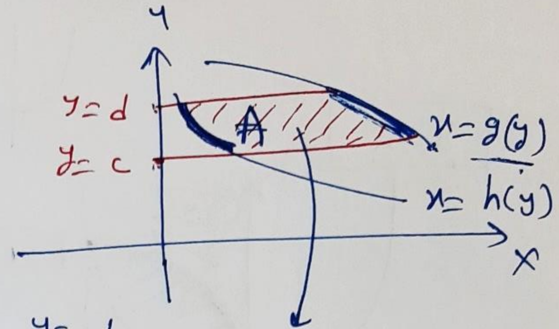
$$= \frac{9}{4} \text{ sq. units.}$$

Area between two Curves (दो वक्रों के मध्य का क्षेत्रफल)



$$\int_{x=a}^{x=b}\left(\begin{matrix}\text{ऊपर वाला}\\ \text{graph}\end{matrix} - \begin{matrix}\text{नीचे वाला}\\ \text{graph}\end{matrix}\right).dx$$

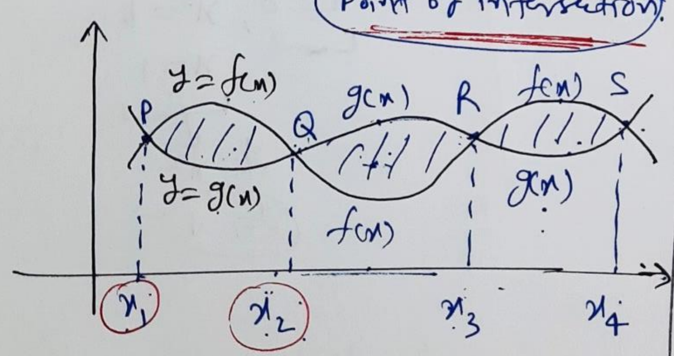
$$\boxed{A = \int_a^b \left(f(x) - g(x)\right).dx}$$



$$\int_{y=c}^{y=d}\left(\begin{matrix}\text{Right}\\ \text{वाला}\end{matrix} - \begin{matrix}\text{left}\\ \text{वाला}\end{matrix}\right).dy$$

$$\boxed{A = \int_c^d \left(g(y) - h(y)\right).dy}$$

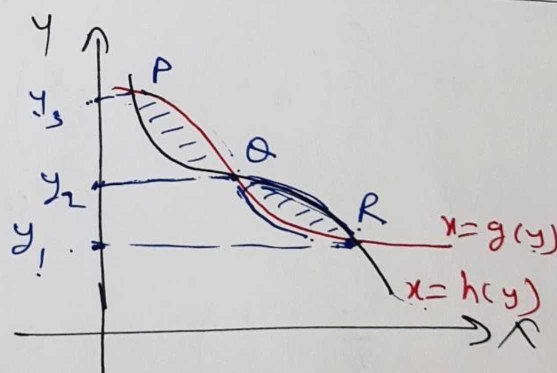
Point of intersection.



$$A = \int_{x_1}^{x_2}\left(f(x) - g(x)\right).dx$$

$$+ \int_{x_2}^{x_3}\left(g(x) - f(x)\right).dx$$

$$+ \int_{x_3}^{x_4}\left(f(x) - g(x)\right).dx$$

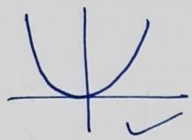


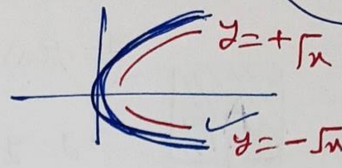
$$A = \int_{y_1}^{y_2}\left(h(y) - g(y)\right).dy$$

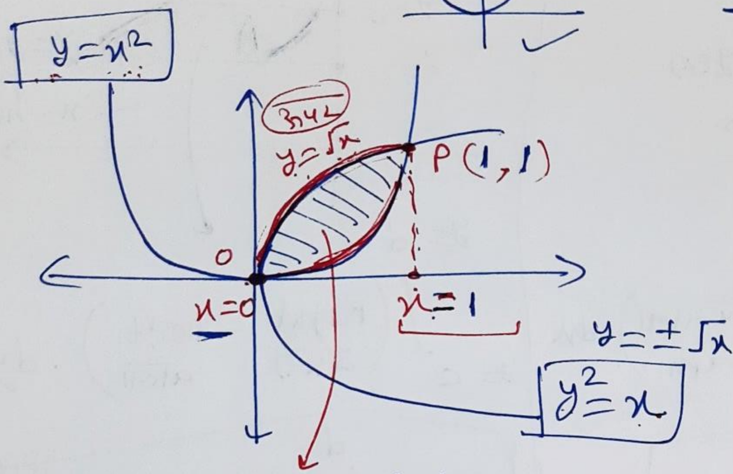
$$+ \int_{y_2}^{y_3}\left(g(y) - h(y)\right).dy$$

e.g. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

$y = \pm\sqrt{x}$







$y^2 = x$ —(1)

$y = x^2$ —(2)

For Point 'P' Solve

$$(x^2)^2 = x$$

$$\Rightarrow x^4 = x$$

$$\Rightarrow x^4 - x = 0$$

$$\Rightarrow x(x^3-1) = 0$$

$$x = 0, \quad x^3 = 1$$

$$x = 1$$

$$y = x^2$$

$$y = 1$$

Required Area $= \displaystyle\int_{x=0}^{x=1} (\sqrt{x} - x^2)\,dx$

terms of x

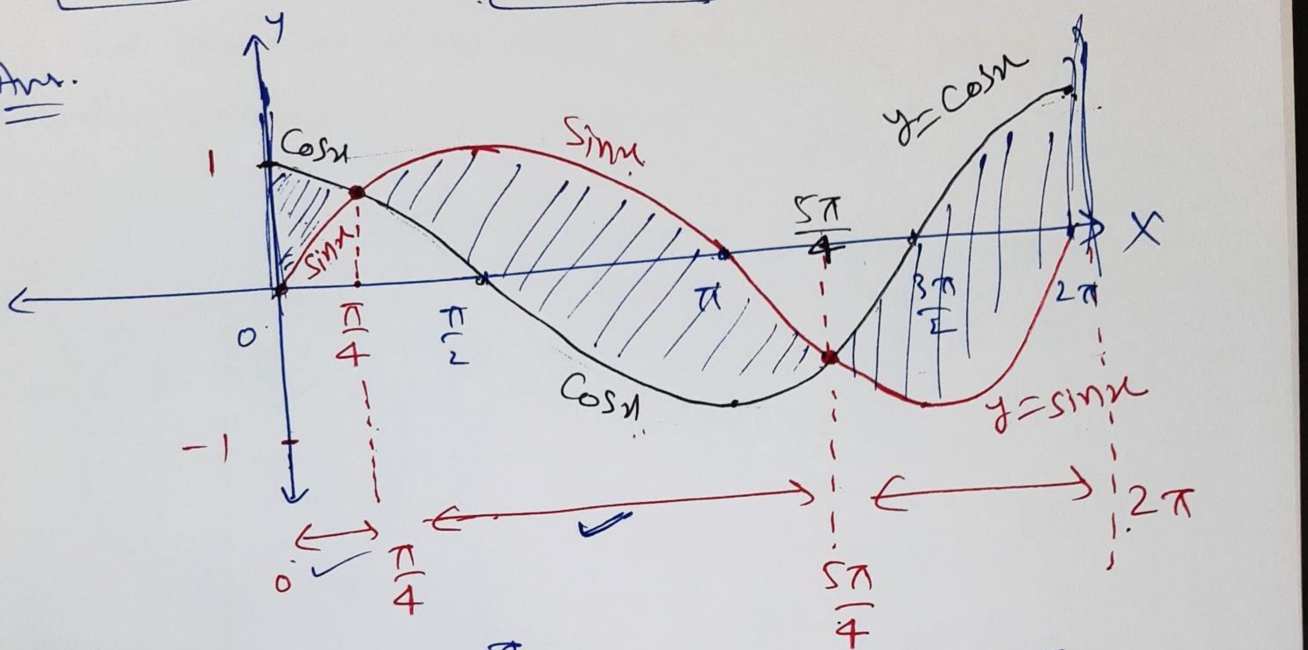
$$= \left(\frac{2}{3}\left(x^{3/2}\right) - \frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0)$$

$$= \frac{1}{3}$$

e.g. The area bounded by the curves $y = \sin x$ and $y = \cos x$ when $0 \le x \le 2\pi$.

Ans.



$$\text{Required Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,.dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,.dx$$

$$+ \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,.dx$$

$$= (\sin x + \cos x)_0^{\pi/4} + (-\cos x - \sin x)_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

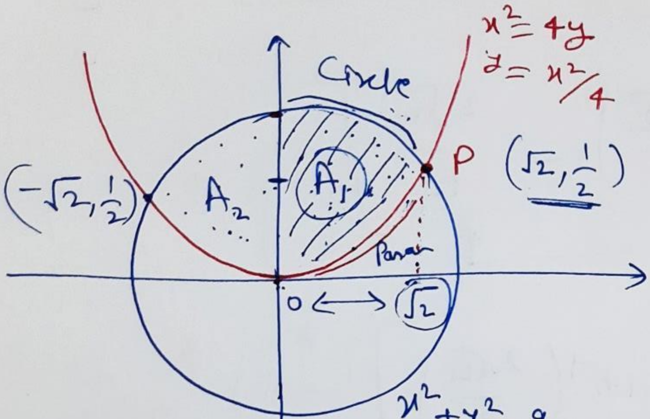
$$+ (\sin x + \cos x)_{\frac{5\pi}{4}}^{2\pi}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - \cancel{1}\right) + \left(+\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$+ \left(0 + \cancel{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

Exercise 8.2 (Area between two Curves)

Q.1 Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$.



Symmetric

$A_1=A_2$

Required Area $=A_1+A_2$

$=2A_1$ terms in x

$=2\int_0^{\sqrt{2}}\left(\sqrt{\frac{9}{4}-x^2}-\frac{x^2}{4}\right).dx$

(ऊपर - नीचे)

Circle - Parabola

$=2\int_0^{\sqrt{2}}\left(\sqrt{\frac{9}{4}-x^2}-\frac{x^2}{4}\right).dx$

$=2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2}+\frac{9/4}{2}\sin^{-1}\left(\frac{x}{3/2}\right)-\frac{x^3}{12}\right]_0^{\sqrt{2}}$

$x^2+y^2=\frac{9}{4}=\left(\frac{3}{2}\right)^2$

$\left[\begin{array}{l}\text{Centre}=(0,0)\\ r=\frac{3}{2}\end{array}\right]$

For Point P

$x^2=4y$, $4x^2+4y^2=9$

$4(4y)+4y^2=9$

$\Rightarrow 4y^2+16y-9=0$

$\Rightarrow 4y^2+18y-2y-9=0$

$\Rightarrow 2y(2y+9)-1(2y+9)=0$

$\Rightarrow (2y-1)(2y+9)=0$

$y=\frac{1}{2}$, $y=-\frac{9}{2}$ ×

$x^2=4y=4\left(\frac{1}{2}\right)=2$

$x^2=2\Rightarrow (x=\pm\sqrt{2})$

Circle $x^2+y^2=\frac{9}{4}$

$\Rightarrow y^2=\frac{9}{4}-x^2$

$\Rightarrow y=\pm\sqrt{\frac{9}{4}-x^2}$

$y=(+)$

$y=(-)$

Required area $= 2A_1$

$$= 2\left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\underbrace{\sqrt{\frac{9}{4}-2}}_{\to\frac{1}{2}} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12} - 0\right]$$

$$= 2\left(\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12}\right)$$

LCM

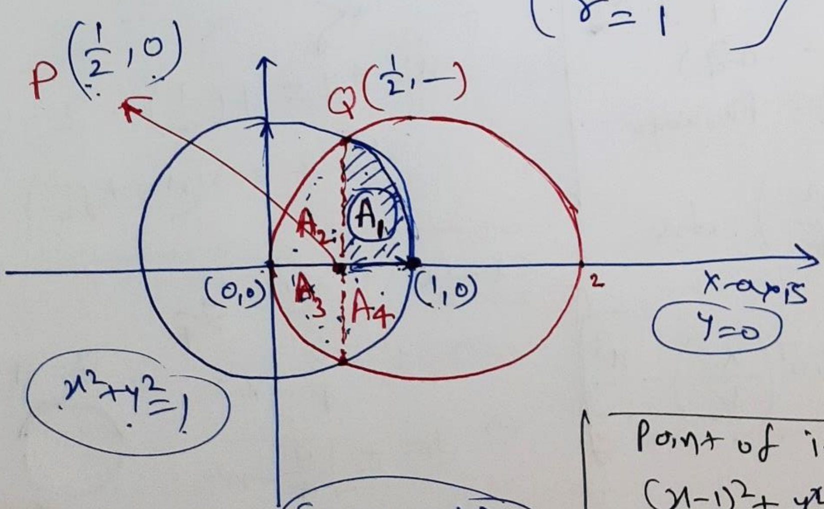
$$= 2\left[\frac{3\sqrt{2}-2\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Q.2 Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

$x^2+y^2=1$ circle $\rightarrow$ [centre $(0,0)$, $r=1$]

$(x-1)^2+y^2=1$ circle [centre $(1,0)$, $r=1$]



Symmetric

$A_1 = A_2 = A_3 = A_4$

Required Area

$= A_1 + A_2 + A_3 + A_4$

$= 4A_1$

Point of intersection $Q \rightarrow \left(\frac{1}{2}, -\right)$

$(x-1)^2 + \cancel{y^2} = \cancel{1} \longrightarrow$

$-\;x^2 + \cancel{y^2} = \cancel{1} \longrightarrow$

$\Rightarrow (x-1)^2 - x^2 = 0 \longrightarrow x = \frac{1}{2}$

Required Area $= 4A_1$

terms in x

$$= 4 \cdot \int_{x=\frac{1}{2}}^{x=1} \left(\sqrt{1-x^2} - 0 \right) . dx$$

$(3x^2 - ~~\text{...}~~)$

$x^2+y^2=1$ $\quad$ $y=0$

$$y^2 = 1-x^2$$

$$y = \pm\sqrt{1-x^2}$$

$$= 4 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right) \right]_{\frac{1}{2}}^{1}$$

$$= 4 \left[\underbrace{\cancel{\frac{1}{2}\sqrt{1-1}}^{\,0} + \frac{1}{2}\underbrace{\sin^{-1}(1)}_{\pi/2}} - \underbrace{\frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}\underbrace{\sin^{-1}\left(\frac{1}{2}\right)}_{\pi/6}} \right]$$

$$= 4\left[\frac{\pi}{4} - \frac{1}{4}\sqrt{\frac{3}{4}} - \frac{1}{2}\times\frac{\pi}{6} \right]$$

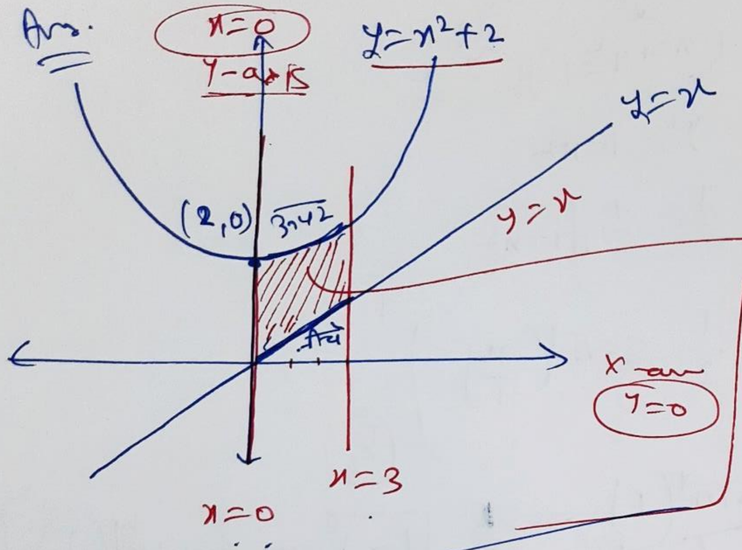
$$= 4\left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} \right] = 4\left[\frac{3\pi-\pi}{12} - \frac{\sqrt{3}}{8} \right]$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

Q.3 Find the area of the region bounded by the curves $\underline{y=x^2+2}$, $\underline{y=x}$, $\underline{x=0}$ and $\underline{x=3}$.

↑ Parabola   line   vertical lines

Ans.



→ Required Area

$$= \int_{x=0}^{x=3} \Big(\underbrace{x^2+2}_{\text{ऊपर}} - \underbrace{x}_{\text{नीचे}}\Big)\,dx$$

$y = x^2+2$ $y = x$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

$$= \left[9 + 6 - \frac{9}{2} - 0\right] = \frac{9}{2} + 6 = \frac{9+12}{2} = \frac{21}{2}$$

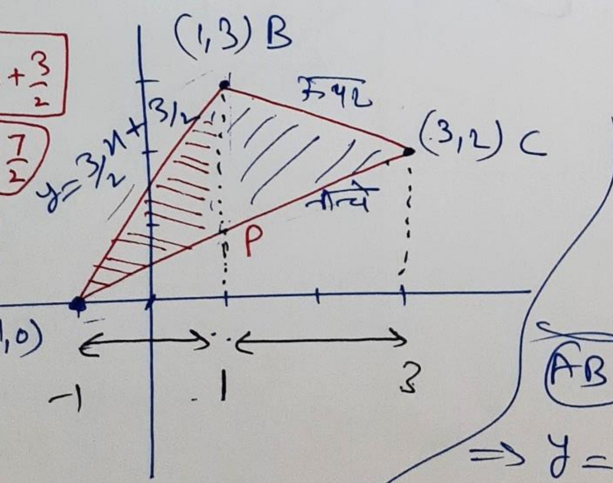
Q.4 Using integration find the area of region bounded by the triangle whose vertices are $(-1,0)$, $(1,3)$ & $(3,2)$

Exactly Plot.

AB $\boxed{y = \frac{3}{2}x + \frac{3}{2}}$

BC $\boxed{y = -\frac{x}{2} + \frac{7}{2}}$

AC $\boxed{y = \frac{x}{2} + \frac{1}{2}}$



Equation (x_2, y_2) (x_1, y_1)

$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

AB $\quad y - 3 = \left(\frac{3-0}{1+1}\right)(x-1)$

$$\Rightarrow y = \frac{3}{2}x - \frac{3}{2} + 3$$

$$\Rightarrow \boxed{y = 3/2\, x + 3/2}$$

Required Area $= ar(\Delta APB) + ar(\Delta BPC)$

$$= \int_{-1}^{1} \left(\frac{3}{2}x + \frac{3}{2} - \frac{x}{2} - \frac{1}{2}\right) \cdot dx + \int_{1}^{3} \left(-\frac{x}{2} + \frac{7}{2} - \frac{x}{2} - \frac{1}{2}\right) \cdot dx$$

$(AB - AP)$ $\quad$ $BC - PC$

$$= \int_{-1}^{1} (x+1) \cdot dx + \int_{1}^{3} (-x+3) \cdot dx$$

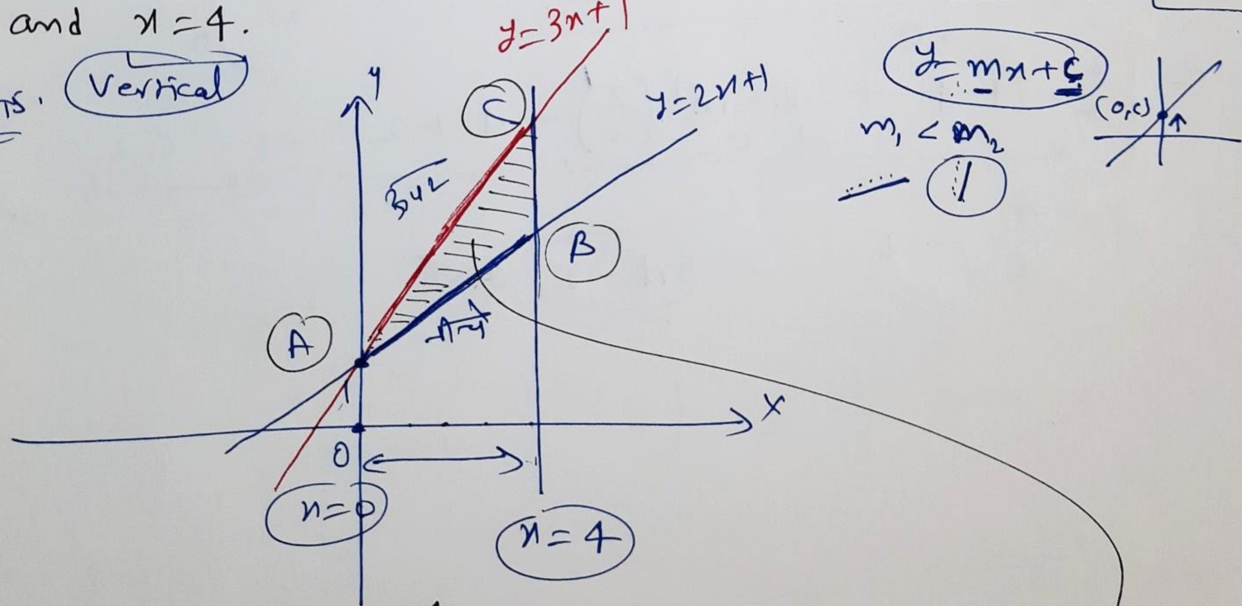
$$= \left(\frac{x^2}{2} + x\right)_{-1}^{1} + \left(-\frac{x^2}{2} + 3x\right)_{1}^{3}$$

$$= \left[\cancel{\frac{1}{2}} + 1 - \cancel{\frac{1}{2}} + 1\right] + \left[-\frac{9}{2} + 9 + \frac{1}{2} - 3\right]$$

$$= 8 - 4 = 4$$

Q.5 Using Integration find the area of the triangle region whose sides have the equations $y = 2x+1$, $y = 3x+1$ and $x = 4$.

Ans. Vertical



Required area $= \int_{0}^{4} (3x+1-2x-1) \cdot dx$
ar(ΔABC)

$$= \int_{0}^{4} x \cdot dx = \left(\frac{x^2}{2}\right)_{0}^{4} = \frac{16}{2} - 0 = 8$$

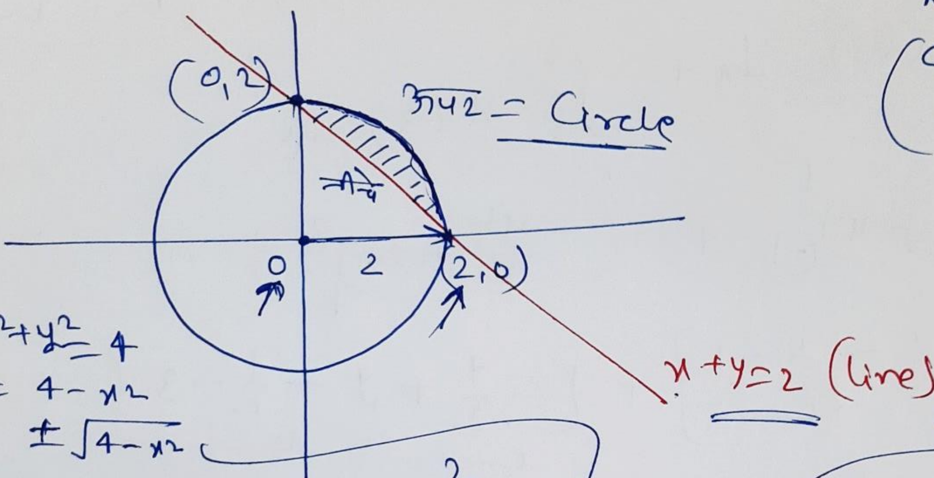
Q.6 Smaller area enclosed by the circle $x^2+y^2=4$ and the line $x+y=2$ is —

(A) $2(\pi-2)$ (B) $\pi-2$ (C) $2\pi-1$ (D) $2(\pi+2)$

Ans.

$x^2+y^2=4=r^2$

(Centre $(0,0)$, $r=2$)



$x^2+y^2=4$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$y=2-x$

$$\text{Required area} = \int_0^2 \left(\sqrt{4-x^2} - 2 + x\right).dx$$

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$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$$= \left[\cancel{\frac{2}{2}\sqrt{4-4}}^{\,0} + 2\sin^{-1}\left(\cancel{\frac{2}{2}}\right) - 4 + 2 - 0\right]$$

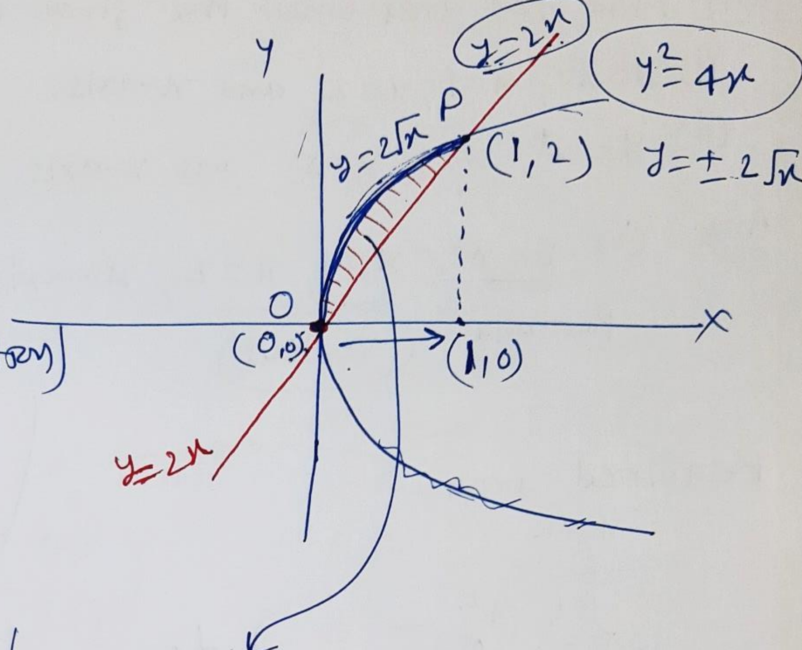
$$= \cancel{2}\times\frac{\pi}{\cancel{2}} - 2 = \pi-2$$

Q.7 Area lying between the curves $y^2 = 4x$ and $y = 2x$ is — (A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

Ans. $y^2 = 4x$ (Parabola)

$y = 2x$

$y = mx + c$



P (Point of intersection)

$y = 2x$ ——

$y^2 = 4x$ ——

$\Rightarrow (2x)^2 = 4x$

$\Rightarrow \cancel{4}x^2 = \cancel{4}x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$x = 0, x = 1$

O P

$y = 2(1)$

$y = 2$

$P(1,2)$

Required Area

$$= \int_0^1 (2\sqrt{x} - 2x)\, dx$$

$y = 2\sqrt{x}$ $y = 2x$

$$= \left(2 \cdot \frac{2}{3} \cdot x^{3/2} - \cancel{2} \cdot \frac{x^2}{\cancel{2}}\right)_0^1$$

$$= \left(\frac{4}{3} \times 1 - 1 - 0\right) = \frac{4}{3} - 1 = \frac{4-3}{3}$$

$$= \frac{1}{3}$$

Miscellaneous Exercise on Chapter 8 ⇒

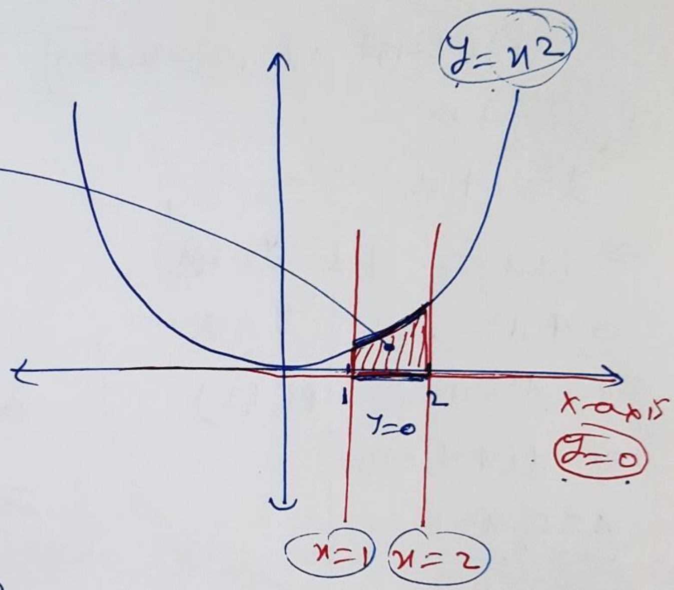
Q.1 Find the area under the given curves and given lines :

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis.

(ii) $y = x^4$, $x = 1$, $x = 5$ and x-axis

Ans. (i) $y = x^2$, $x = 1$, $x = 2$, x-axis

Parabola vertical



Required area

$$= \int_{x=1}^{x=2} (x^2 - 0)\,.dx$$

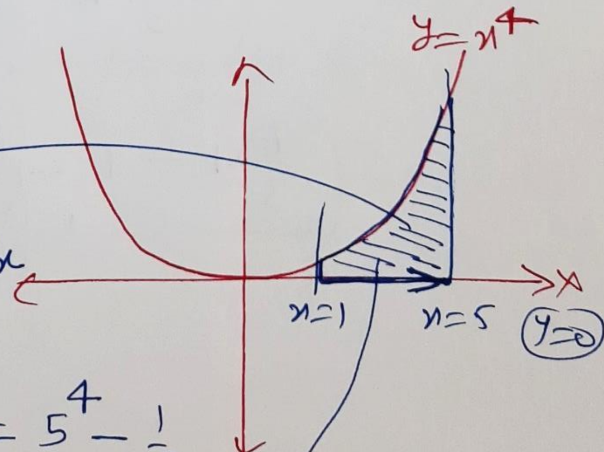
उपर − नीचे

$y = x^2$ $y = 0$

$$= \left(\frac{x^3}{3}\right)_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(ii) $y = x^4$, $x = 1$, $x = 5$, x-axis

vertical



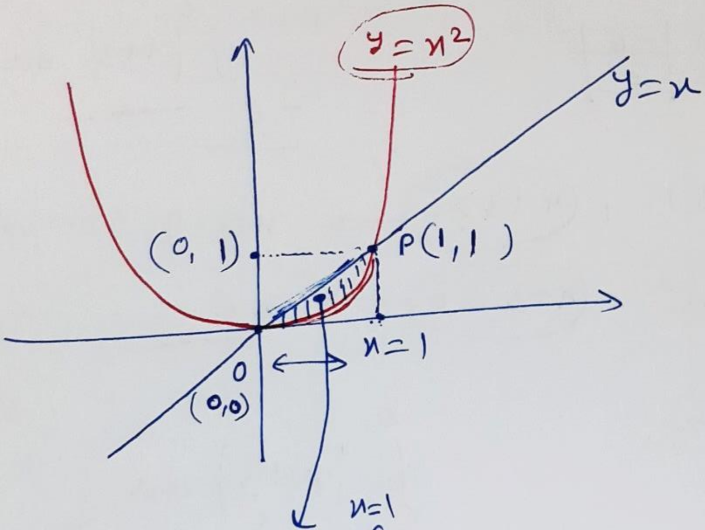
$$\text{Required area} = \int_{x=1}^{x=5} (x^4 - 0)\,.dx$$

उपर − नीचे

$$= \left(\frac{x^5}{5}\right)_1^5 = \frac{5^5}{5} - \frac{1}{5} = 5^4 - \frac{1}{5}$$

$$= 625 - 0.2 = 624.8$$

Q.2 Find the area between the curves $y = x$ and $y = x^2$.



For 'P' (Point of intersection)

$y = x$ —(1)

$y = x^2$ —(2)

$\Rightarrow x = x^2$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow \underbrace{x = 0}$ $\underbrace{\boxed{x = 1}}_{P}$ | $y = 1$

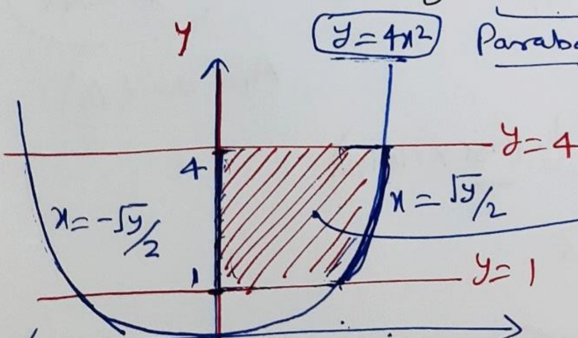
$$\text{Required Area} = \int_{x=0}^{x=1} (x - x^2)\,.\,dx$$

$y = x$ $\quad$ $y = x^2$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0) = \frac{3-2}{6} = \frac{1}{6}$$

Q.3 Find the area of the region lying in the first quadrant and bounded by $\underbrace{y = 4x^2}_{\text{Parabola}}$, $\underbrace{x = 0}_{\text{Y-axis}}$, $\underbrace{y = 1, y = 4}_{\text{Horizontal}}$.



$x = 0$

$y = 4x^2$

$\frac{y}{4} = x^2$

$\pm \frac{\sqrt{y}}{2} = x$

Required Area

$$= \int_{y=1}^{y=4} \left(\frac{\sqrt{y}}{2} - 0\right).\,dy$$

Right − left

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \left(y^{3/2}\right)_1^4$$

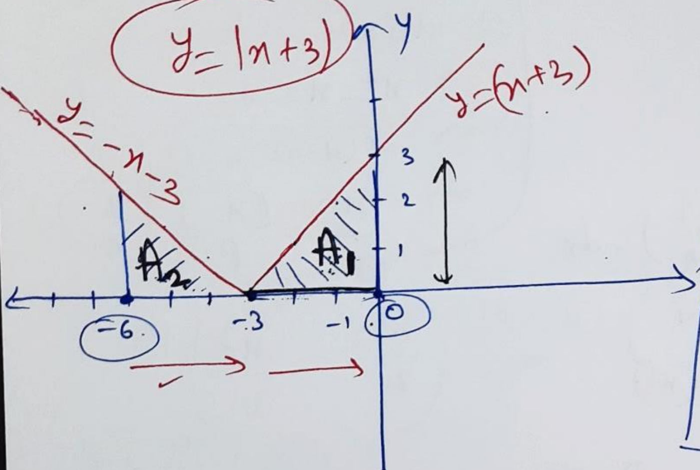
$$= \frac{1}{3}\left[4^{3/2} - 1\right] = \frac{1}{3}(8 - 1)$$

$$= 7/3$$

Q.4 Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3| \, dx$

x	0	1	-1	-2	-3	-4	-5	-6
y	3	4	2	1	0	1	2	3

$$y = |x+3| = \begin{cases} +(x+3), & x+3 \geq 0 \rightarrow x \geq -3 \\ -(x+3), & x+3 < 0 \rightarrow x < -3 \end{cases}$$



$\int_{-6}^{0} |x+3| \, dx$

$x+3=0$

$x=-3$

$$= \int_{-6}^{-3} (-x-3) \, dx + \int_{-3}^{0} (x+3) \, dx$$

$= 9$

$\int_{-6}^{0} |x+3| \, dx$ = Area under the curve $y = |x+3|$ upto x-axis

$A_1 = A_2$

$= A_2 + A_1$

$= 2A_1$ $\qquad A_1 = ar(\Delta)$

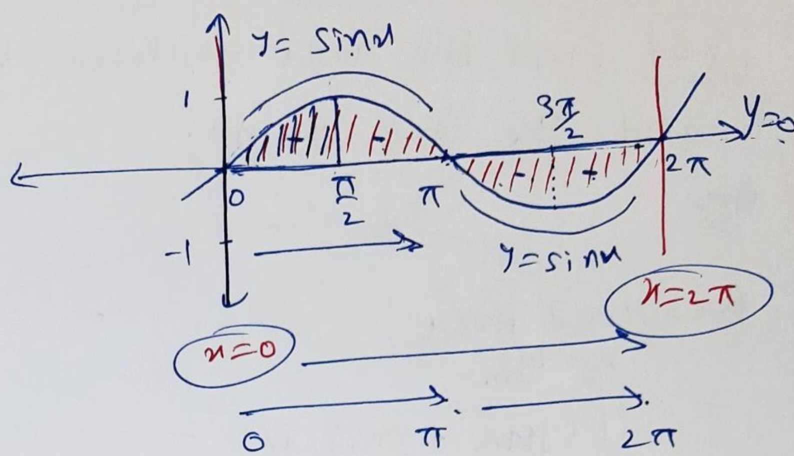
$= \cancel{2} \times \left(\cancel{\frac{1}{2}} \text{ Base} \times \text{Height}\right)$

$= 3 \times 3$

$= 9$ a

Q.5 Find the area bounded by the curve $y = \sin x$ between $x = 0$ & $x = 2\pi$.

Ans



Required area

$$= \int_0^{\pi} (\sin x - 0)\,.\,dx + \int_{\pi}^{2\pi} (0 - \sin x)\,.\,dx$$

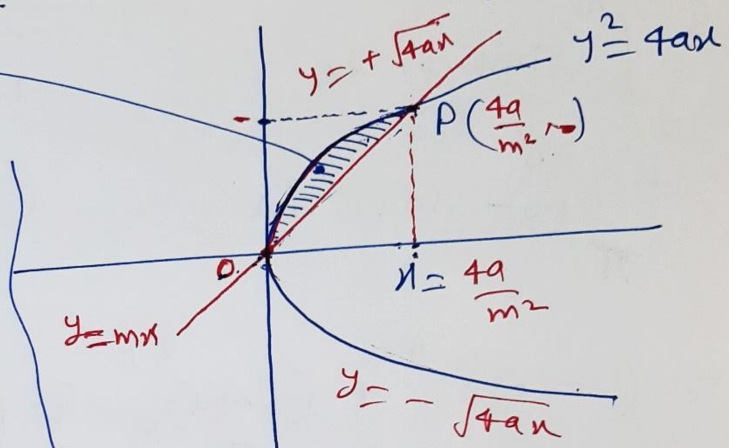
$$= \left(-\cos x\right)_0^{\pi} + \left(\cos x\right)_{\pi}^{2\pi} = \left(-\cos\pi + \cos 0\right) + \left(\cos 2\pi - \cos\pi\right)$$

$$= 1 + 1 + 1 + 1 = 4$$

Miscellaneous Exercise on Chapter 8

Q.6 Find the area enclosed by parabola $y^2 = 4ax$ and the line $y = mx$.

Ans.



Required Area

$$= \int_{x=0}^{x=4a/m^2} \left(\sqrt{4ax} - mx\right).dx$$

Parabola — line

$y^2 = 4ax$ $y = mx$

$y = \pm\sqrt{4ax}$

P (Point of intersection)

$y = mx \;\to\; y^2 = 4ax$

$\Rightarrow (mx)^2 = 4ax$

$\Rightarrow m^2x^2 - 4ax = 0$

$\Rightarrow x(m^2x - 4a) = 0$

$\Rightarrow x = 0,\; x = \dfrac{4a}{m^2}$

$$= \int_0^{4a/m^2} \left(2\sqrt{a}\sqrt{x} - mx\right).dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{3/2} - m\frac{x^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2\right] - [0]$$

$$= \frac{4}{3}\sqrt{a}\cdot\frac{8a\sqrt{a}}{m^3} - \frac{\cancel{m}}{\cancel{2}}\cdot\frac{\overset{8}{\cancel{16}}a^2}{\cancel{m^4}\,m^3} = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

$$= \frac{32a^2 - 24a^2}{3m^3} = \boxed{\frac{8a^2}{3m^3}}$$

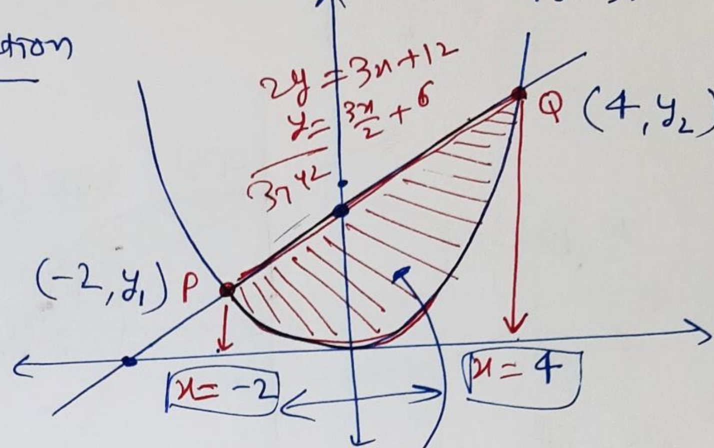
Q.7 Find the area enclosed by the Parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

$y = \frac{3}{4}x^2$

$4y = 3x^2$

Ans.

P & Q → point of intersection



$$2y = 3x + 12$$

$$4y = 3x^2$$

$\Rightarrow 2(2y) = 3x^2$

$\Rightarrow 2(3x + 12) = 3x^2$

$\Rightarrow 6x + 24 = 3x^2$

$\Rightarrow \dfrac{3x^2 - 6x - 24}{3} = \dfrac{0}{3}$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x^2 - 4x + 2x - 8 = 0$

$\Rightarrow x(x-4) + 2(x-4) = 0$

$(x+2)(x-4) = 0$

$\boxed{x = -2}$ $\boxed{x = 4}$

Required area $= \displaystyle\int_{-2}^{4} \Big(\underbrace{\tfrac{3x}{2} + 6}_{\text{ऊपर}} - \underbrace{\tfrac{3}{4}x^2}_{\text{नीचे}}\Big)\, dx$

Line Parabola

$= \left[\frac{3}{2}\cdot\frac{x^2}{2} + 6x - \frac{3}{4}\cdot\frac{x^3}{\cancel{3}}\right]_{-2}^{4}$

$= \left(\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right)_{-2}^{4}$

$= \left(\frac{3}{\cancel{4}}\cdot\cancel{16}^{\,4} + 24 - 16\right) - \left(3 - 12 + 2\right)$

$= 20 + 7 = 27$

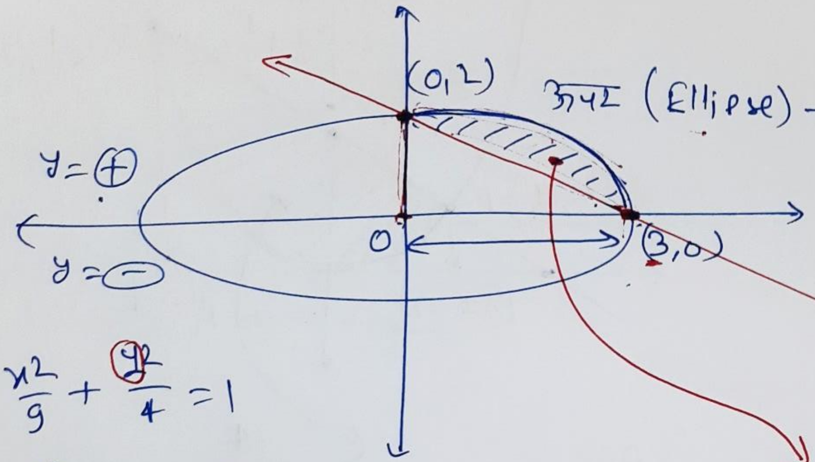
Q.8 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

Intercept.

$\frac{x}{a}+\frac{y}{b}=1$

Ans.

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$



अपर (Ellipse) $\rightarrow y=\frac{2}{3}\sqrt{9-x^2}$

$$\frac{x}{3}+\frac{y}{2}=1$$

$$\frac{y}{2}=1-\frac{x}{3}$$

Line $y=2-\frac{2x}{3}$

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\Rightarrow \frac{y^2}{4}=1-\frac{x^2}{9}$$

$$\Rightarrow y^2=4\left(\frac{9-x^2}{9}\right)$$

$$\Rightarrow \boxed{y=\pm\frac{2}{3}\sqrt{9-x^2}}$$

Required Area

$$=\int_0^3\Big(\underbrace{\frac{2}{3}\sqrt{9-x^2}}_{\text{अपर}}-\underbrace{2+\frac{2x}{3}}_{\text{नीचे (Line)}}\Big)\,dx$$

$$=\left[\frac{2}{3}\left\{\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right\}-2x+\frac{\cancel{2}x^2}{3\cdot\cancel{2}}\right]_0^3$$

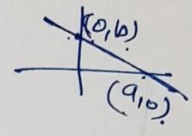
$$=\left[\frac{2}{3}\left\{\frac{3}{2}\sqrt{9-9}+\frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\}-6+3\right]-\left[0\right]$$

$$=3\sin^{-1}(1)-3$$

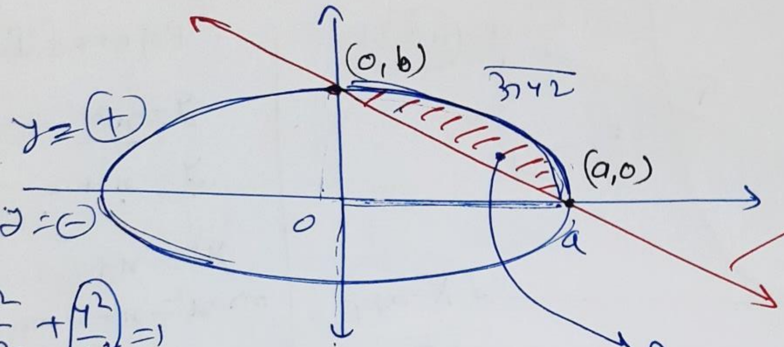
$$=3\left(\frac{\pi}{2}\right)-3$$

$$=3\left(\frac{\pi}{2}-1\right)=3\left(\frac{\pi-2}{2}\right)=\frac{3}{2}(\pi-2)$$

Q.9 Find the area of smaller region bounded by the ellipse $\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}_{\text{Standard Form.}}$ and the line $\underbrace{\frac{x}{a}+\frac{y}{b}=1}_{\text{Intercept Form}}$.



Ans.



$$\frac{x}{a}+\frac{y}{b}=1$$
$$y=\left(b-\frac{bx}{a}\right)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$
$$\Rightarrow y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$
$$\Rightarrow \boxed{y=\pm\frac{b}{a}\sqrt{a^2-x^2}}$$

Required Area

$$=\int_0^a\left(\underbrace{\frac{b}{a}\sqrt{a^2-x^2}}_{\text{ऊपर}}-\underbrace{b+\frac{bx}{a}}_{\text{नीचे}}\right).dx$$

(ऊपर − नीचे)

$$=\left[\frac{b}{a}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\}-bx+\frac{bx^2}{a\,2}\right]_0^a$$

$$=\left[\frac{b}{a}\left\{\cancel{\frac{a}{2}\sqrt{a^2-a^2}}^{\,0}+\frac{a^2}{2}\sin^{-1}(1)\right\}-ba+\frac{ba^2}{2a}\right]-[0]$$

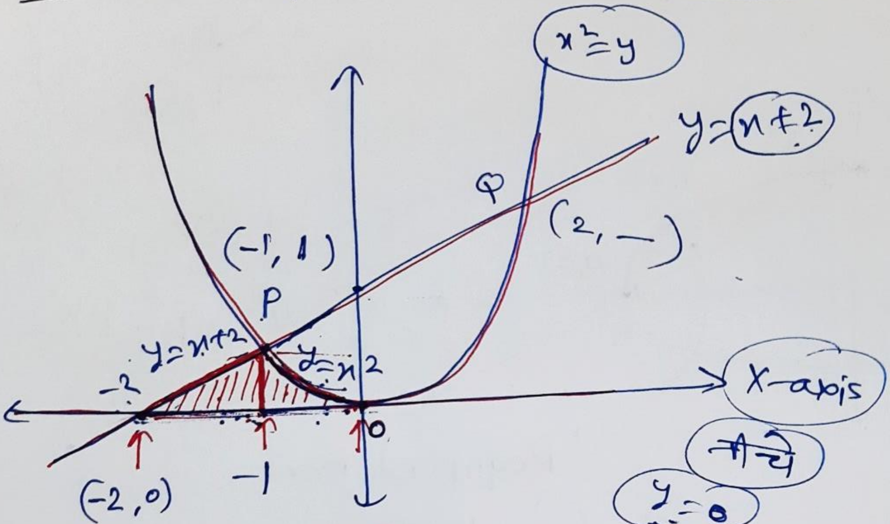
$$=\frac{ab}{2}\cdot\frac{\pi}{2}-ba+\frac{ba}{2}=\frac{\pi ab}{4}-\frac{ba}{2}$$

$$=ab\left\{\frac{\pi}{4}-\frac{1}{2}\right\}=ab\left\{\frac{\pi-2}{4}\right\}=\frac{ab}{4}(\pi-2)$$

$\boxed{Q.10}$ Find the area of the region enclosed by the Parabola $x^2 = y$, the line $y = x+2$ and the x-axis.

Ans.

Ψ $\qquad$ $y = mx + c$



$-2 \longrightarrow 0$

$-2 \longrightarrow -1 \longrightarrow 0$

For P (Point of Intersection)

$y = x^2 \quad \text{---} (1)$

$y = x + 2 \quad \text{---} (2)$

$x^2 = x + 2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x-2)(x+1) = 0$

$\boxed{x = 2, -1}$

Required Area $= \displaystyle\int_{-2}^{-1} (x+2 - 0).\,dx + \int_{-1}^{0} (x^2 - 0).\,dx$

$= \left(\dfrac{x^2}{2} + 2x\right)_{-2}^{-1} + \left(\dfrac{x^3}{3}\right)_{-1}^{0}$

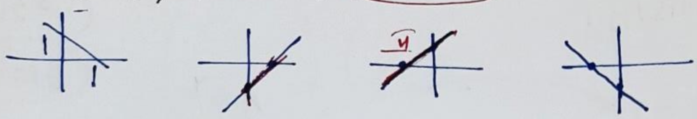
$= \left(\dfrac{1}{2} - 2\right) - (2 - 4) + (0) - \left(\dfrac{-1}{3}\right)$

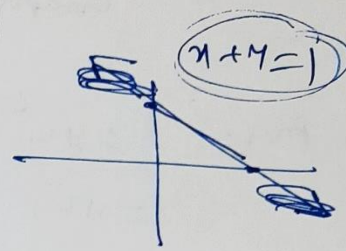
$= \dfrac{1}{2} - \cancel{2} + \cancel{2} + \dfrac{1}{3} = \dfrac{3+2}{6} = \dfrac{5}{6}$

Miscellaneous Exercise on Chapter (8)

Q.11 Using the method of integration find the area bounded by the curve $|x|+|y|=1$. [Hint: The required region is bounded by lines $x+y=1$, $x-y=1$, $-x+y=1$ and $-x-y=1$]

Ans.

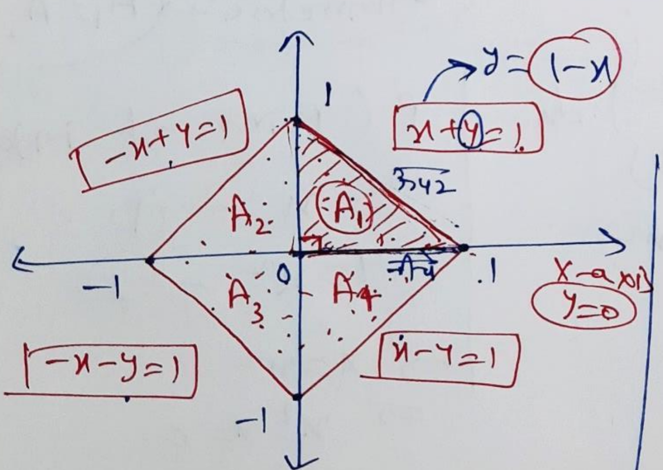




$\boxed{|x|+|y|=1}$

Quadrant wise

	(x, y)	
I-Quad. $\rightarrow$	$(+,+)$ $\rightarrow$	$\boxed{x+y=1}$
II-Quad. $\rightarrow$	$(-,+)$ $\rightarrow$	$\boxed{-x+y=1}$
III-Quad. $\rightarrow$	$(-,-)$ $\rightarrow$	$\boxed{-x-y=1}$
IV-Quad. $\rightarrow$	$(+,-)$ $\rightarrow$	$\boxed{x-y=1}$



Symmetric

$(A_1 = A_2 = A_3 = A_4)$

Required Area

$= A_1 + A_2 + A_3 + A_4$

$= 4(A_1)$

$= 4 \cdot \int_{x=0}^{x=1} \left(\underset{\text{उपर}}{(1-x)} - \underset{\text{नीचे}}{0} \right) \cdot dx$

$= 4 \int_0^1 (1-x) \cdot dx$

$= 4\left(x - \frac{x^2}{2}\right)_0^1$

$= 4\left(1 - \frac{1}{2} - 0\right) = 4 \times \frac{1}{2}$

$= 2$ Sq. units.

Q.12 Find the area bounded by curves

$\{(x,y): \underbrace{y \geq x^2} \text{ and } \underbrace{y = |x|}\}$.

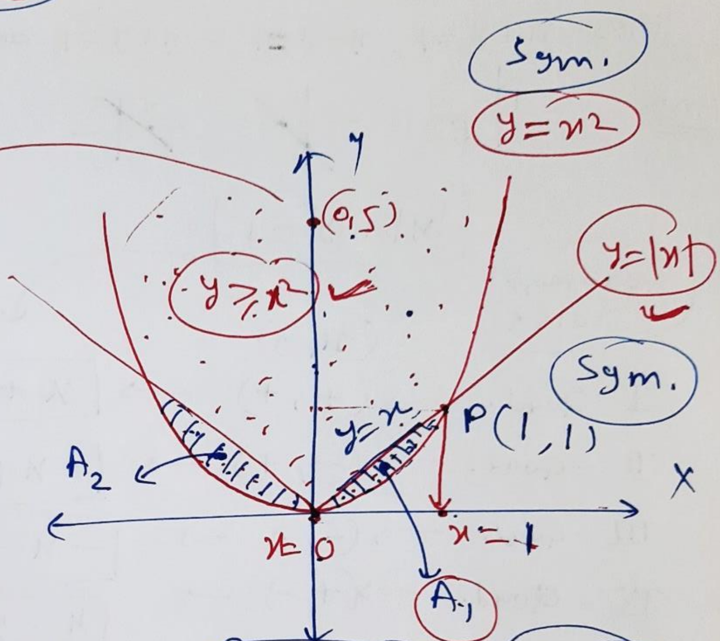
$y = x^2$ Graph ← Point Satisfy → Hit & Trial.

Ans. $y \geq x^2$ ← $(0,5)$

$y = x^2$

$\boxed{5 \geq 0}$

True



Symmetric → $A_1 = A_2$

Required Area

$= A_1 + A_2$

$= 2 A_1$

$= 2 \cdot \int_{x=0}^{x=1} (x - x^2) \cdot dx$

$(\cancel{3x^2} - \cancel{4x})$

$y = x$ $y = x^2$

$= 2 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$

$= 2 \left(\frac{1}{2} - \frac{1}{3}\right)$

$= \cancel{2}\left(\frac{3-2}{\cancel{6}_3}\right) = \frac{1}{3}$

P (Point of intersect)

$y = x$ —— (1)

$y = x^2$ —— (2)

$\Rightarrow x = x^2$

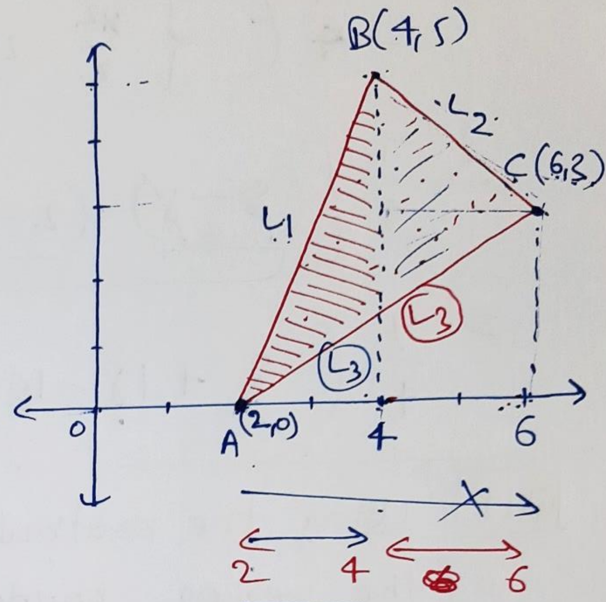
$\Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 0, x = 1$

Q.13 Using the method of integration find the area of triangle ABC, coordinates of whose vertices are A(2,0), B(4,5) and C(6,3).

Ans. $\boxed{(y-y_1) = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)}$



L_1 : AB

$$(y-0) = \left(\frac{5}{2}\right)(x-2)$$

$$\Rightarrow \boxed{y = \frac{5}{2}x - 5} : L_1$$

L_2 : BC $(y-3) = \left(\frac{\cancel{2}^{\,1}}{-\cancel{2}_{\,1}}\right)(x-6)$

$$\Rightarrow y = -x + 6 + 3$$

$$\Rightarrow \boxed{y = -x + 9} \rightarrow L_2$$

L_3 : AC $(y-0) = \left(\frac{3}{4}\right)(x-2)$

$$\Rightarrow \boxed{y = \frac{3x}{4} - \frac{3}{2}} \rightarrow L_3$$

$$ar(\Delta ABC) = \int_2^4 (L_1 - L_3)\,.dx + \int_4^6 (L_2 - L_3)\,.dx$$

$$= \int_2^4 \left(\frac{5}{2}x - 5 - \frac{3x}{4} + \frac{3}{2}\right).dx + \int_4^6 \left(-x + 9 - \frac{3x}{4} + \frac{3}{2}\right).dx$$

$$= \int_2^4 \left(\frac{7x}{4} - \frac{7}{2}\right).dx + \int_4^6 \left(-\frac{7x}{4} + \frac{21}{2}\right).dx$$

$$= \frac{7}{4}\int_2^4 (x-2).dx + \frac{7}{4}\int_4^6 (-x+6).dx$$

$$ar(\Delta ABC) = \left(\frac{7}{4}\right)\int_2^4 (x-2).dx + \left(\frac{7}{4}\right)\int_4^6 (-x+6).dx$$

$$= \frac{7}{4}\left[\left\{\frac{x^2}{2} - 2x\right\}_2^4 + \left\{-\frac{x^2}{2} + 6x\right\}_4^6\right]$$

$$= \frac{7}{4}\left[\underbrace{(\cancel{8}-\cancel{8})-(2-4)} + \underbrace{(-18+36)-(-8+24)}\right]$$

$$= \frac{7}{4}(2+18-16) = \frac{7}{\cancel{4}}(\cancel{4}) = 7$$

Q.14 Using the method of integration find the area of the region bounded by lines:

$2x+y=4 \rightarrow L_1$

$3x-2y=6 \rightarrow L_2$

$x-3y+5=0 \rightarrow L_3$

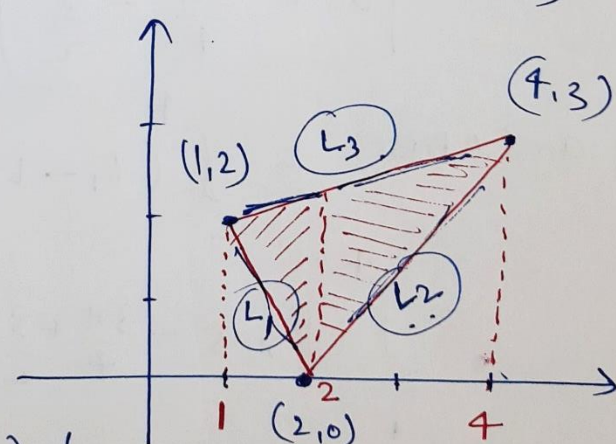
Ans.

L_1, L_2	L_2, L_3	L_3, L_1	
$(2,0)$	$(4,3)$	$(1,2)$	← Point of intersection (vertices)

$L_1 \rightarrow 4-2x = y$

$L_2 \rightarrow y = \frac{3}{2}x - 3$

$L_3 \rightarrow y = \frac{x}{3} + \frac{5}{3}$



$1 \leftrightarrow 2 \leftrightarrow 4$

Required area $= ar(\Delta)$

$$= \int_1^2 (L_3 - L_1).dx + \int_2^4 (L_3 - L_2).dx$$

$$= \int_1^2 \left(\frac{x}{3} + \frac{5}{3} - 4 + 2x\right).dx + \int_2^4 \left(\frac{x}{3} + \frac{5}{3} - \frac{3x}{2} + 3\right).dx$$

$$= \int_1^2 \left(\frac{7x}{3} - \frac{7}{3}\right) \cdot dx + \int_2^4 \left(-\frac{7x}{6} + \frac{14}{3}\right) \cdot dx$$

$$= \frac{7}{3}\left[\int_1^2 (x-1) \cdot dx + \int_2^4 \left(-\frac{x}{2} + 2\right) \cdot dx\right]$$

$$= \frac{7}{3}\left[\left(\frac{x^2}{2} - x\right)_1^2 + \left(-\frac{x^2}{4} + 2x\right)_2^4\right]$$

$$= \frac{7}{3}\left[\left(\cancel{2} - \cancel{2}\right) - \left(\frac{1}{2} - 1\right) + (-4+8) - (-1+4)\right]$$

$$= \frac{7}{3}\left[-\frac{1}{2} + 1 + 4 - 3\right] = \cancel{\frac{7}{3}}\left(\frac{3}{2}\right) = \frac{7}{2}$$

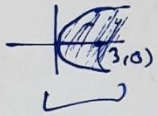
Q.15 Find the area of the region

$$\{(x, y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}$$

$x^2 + y^2 = \frac{9}{4}$

$r = \frac{3}{2}$ $(0,0)$

Ans. $y^2 = 4x$

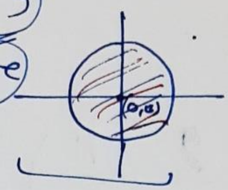


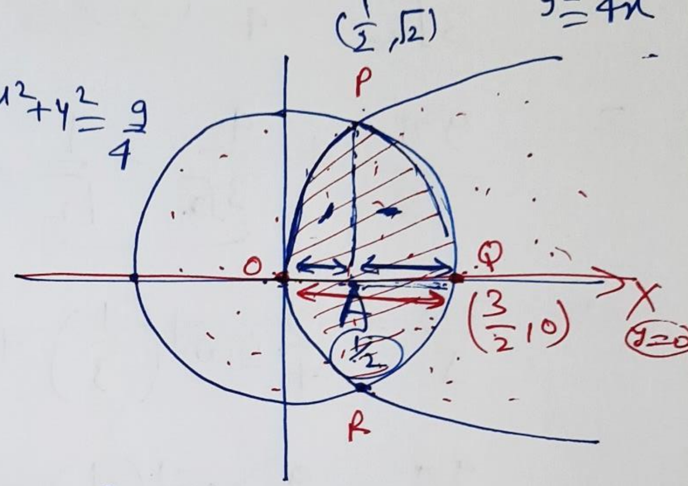
$0 \le 4 \times 3$

$0 \le 12$ True

$0 \le 9$

True





Required area

$= ar(OPQRO)$

$= 2 \times \boxed{ar(OPQO)}$

$$= 2\left[\int_0^{1/2} (2\sqrt{x} - 0) \cdot dx + \int_{1/2}^{3/2} \left(\sqrt{\frac{9}{4} - x^2} - 0\right) \cdot dx\right]$$

P → Point of intersection

$y^2 = 4x$ $4x^2 + 4y^2 = 9$

$\Rightarrow x = \frac{1}{2}$, $\cancel{x = -\frac{9}{2}}$

$$= 2\left[\int_0^{1/2} 2\sqrt{x} \cdot dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2} \cdot dx\right]$$

$$= 2\left[2 \cdot \frac{2}{3} \cdot \left(x^{3/2}\right)_0^{1/2} + \left(\frac{x}{2}\sqrt{9/4 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right)_{1/2}^{3/2}\right]$$

$$= 2\left[\frac{4}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + \frac{3}{4}\sqrt{\frac{9}{4} - \frac{9}{4}} + \frac{9}{8}\sin^{-1}\left(\frac{2}{3} \cdot \frac{3}{2}\right) - \frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

required area $= 2\left[\frac{\cancel{4}^{2}}{3}\left(\frac{1}{\cancel{2}\sqrt{2}}\right) + \frac{9}{8}\sin^{-1}(1) - \frac{1}{4}\sqrt{\frac{\cancel{8}2}{\cancel{4}}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$

$$= \frac{4}{3\sqrt{2}} + \frac{9}{4}\left(\frac{\pi}{2}\right) - \frac{1}{\cancel{2}}\frac{\cancel{\sqrt{2}}^{1}}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9\pi}{8} + \underbrace{\frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right).$$

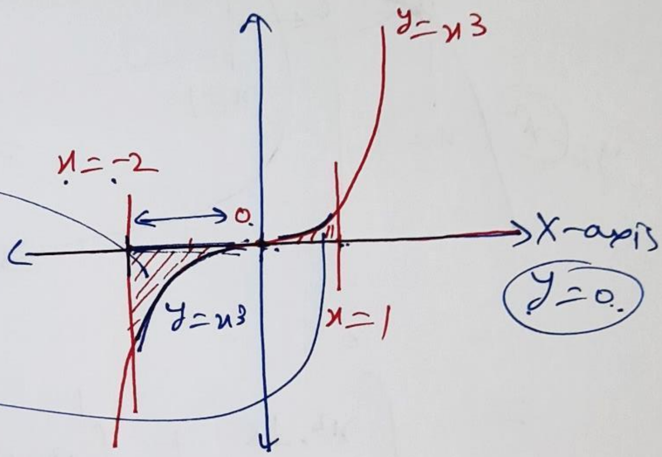
$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{4-3}{3\sqrt{2}}$$

$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}$$

Miscellaneous Exercise on chapter 8

Q.16 Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is —

(A) -9 (B) $\frac{-15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$.



Required area

$$= \int_{-2}^{0} (0 - x^3)\,dx + \int_{0}^{1} (x^3 - 0)\,dx$$

$$= \left(-\frac{x^4}{4}\right)_{-2}^{0} + \left(\frac{x^4}{4}\right)_{0}^{1} = \left[(0) - (-4)\right] + \left(\frac{1}{4} - 0\right)$$

$$= 4 + \frac{1}{4} = \frac{17}{4}$$

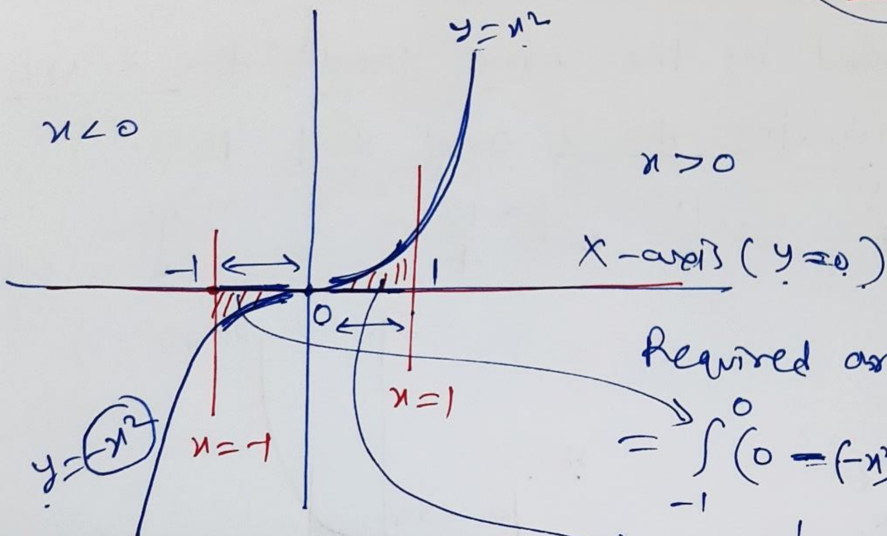
Q.17 The area bounded by the curve $\boxed{y = x|x|}$, x-axis and the ordinates $x = -1$ & $x = 1$, is given by —

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$y = x|x| = \begin{cases} x(x), & x > 0 \\ x(-x), & x < 0 \end{cases} = \begin{cases} x^2, & x > 0 \\ -x^2, & x < 0 \end{cases}$$

$y = x|x| = \begin{cases} x^2, & x>0 \\ -x^2, & x<0 \end{cases}$, x-axis, $x=-1$, $x=1$



Required area

$= \int_{-1}^{0} (0-(-x^2)) \cdot dx$

$+ \int_{0}^{1} (x^2-0) \cdot dx$

$= \int_{-1}^{0} x^2 \cdot dx + \int_{0}^{1} x^2 \cdot dx = \left(\frac{x^3}{3}\right)_{-1}^{0} + \left(\frac{x^3}{3}\right)_{0}^{1}$

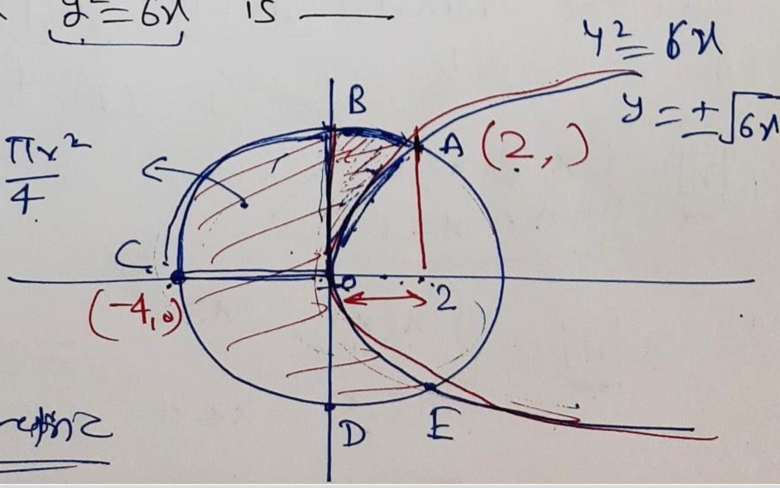
$= \left(0+\frac{1}{3}\right) + \left(\frac{1}{3}-0\right) = \frac{2}{3}$

Q.18 The area of the circle $\boxed{x^2+y^2=16}$, exterior to the parabola $y^2=6x$ is ——

$x^2+y^2=16=4^2$

$(0,0)$, $r=4$

$4\pi = \frac{\pi r^2}{4}$



Ex. interior

Symmetric

required area = ar(OABCDEO) = 2 · ar(OABCO)

$= 2\left(ar(OBCO) + ar(OABO)\right)$

Quadrant 

$\frac{\pi r^2}{4} = \frac{\pi 4^2}{4} = 4\pi$

$\int_0^2 \left(\sqrt{16-x^2} - \sqrt{6x}\right) . dx$ (Circle − Parabola)

$\left(\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right) - \sqrt{6}\,\frac{2}{3}\,x^{3/2}\right)_0^2$

$= \left(2\sqrt{3} + 8\sin^{-1}\left(\frac{1}{2}\right) - \sqrt{6}\cdot\frac{2}{3}\cdot 2\sqrt{2}\right)$

$= \left(2\sqrt{3} - \frac{\sqrt{2}\cdot\sqrt{3}\cdot 4\sqrt{2}}{3} + 8\cdot\frac{\pi}{6}\right)$

$= \left(-\frac{2\sqrt{3}}{3} + \frac{4\pi}{3}\right)$

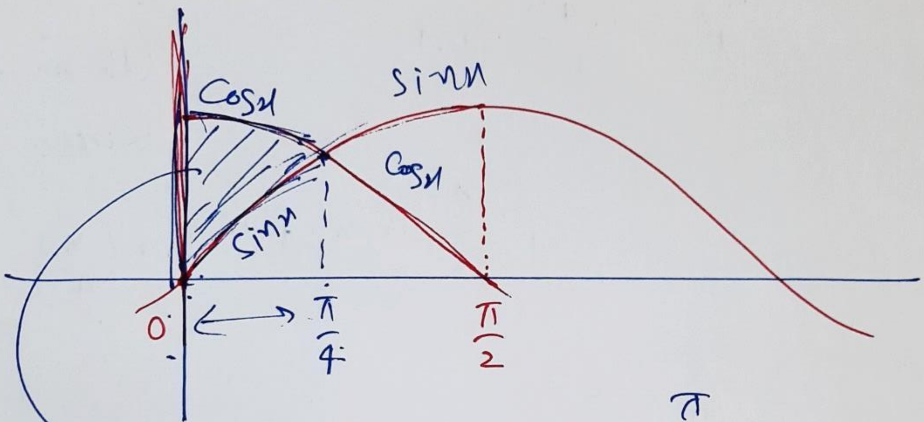
$= 2\left(4\pi - \frac{2\sqrt{3}}{3} + \frac{4\pi}{3}\right)$

$= 2\left(\frac{12\pi - 2\sqrt{3} + 4\pi}{3}\right) = \frac{2}{3}\left(16\pi - 2\sqrt{3}\right)$

$= \frac{4}{3}\left(8\pi - \sqrt{3}\right)$

Q.19 The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$, when $0 \le x \le \frac{\pi}{2}$ is–

(A) $2(\sqrt{2}-1)$ (B) $\sqrt{2}-1$ (C) $\sqrt{2}+1$ (D) $\sqrt{2}$



Required area $= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$

$= \left(\sin x + \cos x\right)_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)$

$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

$2 = \sqrt{2} \times \sqrt{2}$